ON THE QUADRATIC FUNCTION AGAIN

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ABSTRACT

The level of mastering of mathematical knowledge is strongly dependent on the skill to apply this knowledge in problem solving. This means that students should be capable not only of reproductive, but also of creative thinking. What is important for the purpose is a deep entering into the essence of the particular problem under consideration which allows an individual discovery of the appropriate way to the solution. The consecutive stages include applications of theoretical and practical facts, methods, algorithms, ideas, etc. Combinations of ideas and theoretical facts are of great importance, too. The present paper is dedicated to the problem in question directed to quadratic functions. The proposed didactical set aims at training and enhancing the possibilities of a successful application to various mathematical topics.

INTRODUCTION

Quadratic functions, together with linear ones are quite well presented in the Bulgarian school curriculum. The textbooks and the corresponding supporting materials contain a lot of beautiful problems on quadratic function properties, Viete formulae, distribution of roots and s. o. Sometimes the didactical set in the sequel does not mention quadratic functions, quadratic equations or inequalities explicitly at all. The idea is to discover corresponding models including the listed notions. Of course, other approaches in the modeling are possible but they are not discussed in the paper.

A SET OF PROBLEMS

The first two problems are connected with number systems.

Problem 1. Solve the equations:

a) $111_{(x)} = 273$; b) $321_{(x)} + 123_{(x+1)} = 95$.

Solution: In this problem the approach is to represent the numbers on the right hand side as polynomials using the base of the corresponding number system as a variable. Thus:

a) $x^2 + x + 1 = 273$, and we get the quadratic equation $x^2 + x - 272 = 0$ with roots $x_1 = -17$ and $x_2 = 16$. Only $x_2 = 16$ is a solution (the integer x should be greater than 1);

b) $3x^2 + 2x + 1 + (x+1)^2 + 2(x+1) + 3 = 95$, and now the quadratic equation $2x^2 + 3x - 44 = 0$ has roots $x_1 = -5{,}5$ and $x_2 = 4$. Only $x_2 = 16$ is a solution for the same reasons as in a).

The same approach could be applied to the next problem

Problem 2. Find the number system in which the equality $23 = 212_{(x)}$ holds true.

Answer: The base of the number system is equal to 3.

The idea of the solutions of the next two problems is to use the fact that if a quadratic equation has real roots, then its discriminant is not negative.

Problem 3. Real numbers x and y satisfy the relation $y = x + \frac{4}{x-1}$. What are the possible values of y?

Solution: Rewrite the relation in the form $x^2 - (y+1) + y + 4 = 0$ and consider it as a quadratic equation with respect to x. Since x and y are real, we get $D = y^2 - 2y - 15 \geq 0$. From here, $y \in (-\infty;-3] \cup [5;+\infty)$.

Problem 4. Find all real numbers x and y which satisfy the relation $x^2 + 4y^2 - 6x + 8y + 9 = 0$.

Hint. The relation is considered as a quadratic equation first with respect to x and then with respect to y. Thus, $x \in [1;5]$ and $y \in [-2;0]$.

Modeling by quadratic inequalities is applied to the next five problems.

Problem 5. Prove that if the lengths of the sides of a triangle are in a geometric progression, then the quotient of the progression is in the interval

$$\left(\frac{\sqrt{5}-1}{2};\frac{\sqrt{5}+1}{2}\right).$$

Solution: Denote the lengths of the sides of the triangle by a, aq, aq^2, where a is the first member of the progression, and q is the quotient. The triangle inequality implies the following system: $\left|\begin{array}{l} a+aq^2 > aq \\ aq+aq^2 > a \\ a+aq > aq^2 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} q^2-q-1<0 \\ q^2-q+1>0 \\ q^2+q-1>0 \end{array}\right.$. From here we get the solution $q \in \left(\frac{\sqrt{5}-1}{2};\frac{\sqrt{5}+1}{2}\right)$.

Problem 6. Given are real numbers p and q, which are not equal to 0. Prove that $\frac{p^2}{q^2}+\frac{q^2}{p^2}-5\left(\frac{p}{q}+\frac{q}{p}\right)+9\geq 0$. When does the equality hold true?

Solution: Denote $\frac{p}{q}+\frac{q}{p}=x$. Then $\frac{p^2}{q^2}+\frac{q^2}{p^2}=x^2-2$ and it follows that $x^2-5x+7\geq 0$. The latter is verified for all x and the equality is not possible.

Problem 7. Find the values of the real parameter k, for which the inequality $\frac{p^2}{q^2}+\frac{q^2}{p^2}+k\left(\frac{p}{q}+\frac{q}{p}\right)+11>0$ holds true for all real numbers p and q, different from 0.

Solution: Approaching in the same manner as in the previous problem, we get the quadratic inequality $x^2+kx+9>0$. The condition $k^2-36<0$ for the discriminant implies that $k\in(-6;6)$.

Problem 8. Given are such real numbers p, q and r that $p+q+r=0$ and $pq+qr+rp=-3$. Prove that the given numbers are in the interval $[-2;2]$.

Solution: The given relations are symmetric with respect to p, q and r, which means that it is enough to check the assertion for one of them only. From the first relation we get $r=-p-q$ and we substitute it in the second one. Consequently, $q^2+pq+p^2-3=0$. Considering this as a quadratic equation with respect to q, we obtain $p^2-4(p^2-3)\geq 0 \Leftrightarrow p^2-4\leq 0$, i. e. $p\in[-2;2]$.

Problem 9. Given are real numbers x, y, z such that $x+y+z=5$ and $xy+yz+zx=3$. Prove that $-1 \le x \le \frac{13}{3}$, $-1 \le y \le \frac{13}{3}$ and $-1 \le z \le \frac{13}{3}$.

Solution: Rewrite the conditions in the form $x+y=5-z$ and $xy=3-z(x+y)=3-z(5-z)=z^2-5z+3$. Now apply the reverse Viete theorem. It follows that x and y are the roots of the quadratic equation $u^2-(5-z)u+z^2-5z+3=0$, assuming that z is a parameter. The condition for real roots gives $-3z^2+10z+13 \ge 0$ and from here $z \in \left[-1; \frac{13}{3}\right]$, i.e. $-1 \le z \le \frac{13}{3}$. The inequalities for x and y could be proved in a similar way.

Application of quadratic equations to trigonometry problems is illustrated by the next three examples.

Problem 10. Show that a triangle with angles α, β, γ is equilateral iff $\cot g\alpha + \cot g\beta + \cot g\gamma = \sqrt{3}$.

Solution: By $\cot g\gamma = -\cot g(\alpha+\beta) = -\dfrac{\cot g\alpha . \cot g\beta - 1}{\cot g\alpha + \cot g\beta}$ and the condition we get $\cot g\alpha + \cot g\beta - \dfrac{\cot g\alpha . \cot g\beta - 1}{\cot g\alpha + \cot g\beta} = \sqrt{3}$. Set $\cot g\alpha = x$ and $\cot g\beta = y$. Then $x + y - \dfrac{xy-1}{x+y} = \sqrt{3}$. This equation is equivalent to $x^2+(y-\sqrt{3})x+y^2-\sqrt{3}y+1=0$, which could be considered as a quadratic equation with respect to x. The discriminant should not be negative, i.e. $D=-(\sqrt{3}y-1)^2 \ge 0$. Thus, $\sqrt{3}y-1=0$, i.e. $y=\frac{\sqrt{3}}{3}$. From here, $\cot g\beta = \frac{\sqrt{3}}{3}$ and consequently $\beta = \frac{\pi}{3}$. Analogously, $x=\frac{\sqrt{3}}{3}$ and $\alpha=\frac{\pi}{3}$. This means that the triangle is equilateral.

Problem 11. Show that a triangle with angles α, β, γ is equilateral iff $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -\frac{3}{2}$.

Solution: Rewrite the condition as $2\cos^2\gamma - 2\cos(\alpha-\beta).\cos\gamma + \frac{1}{2} = 0$ and denote $\cos\gamma = x$, $x \in (-1;1)$ and $\cos(\alpha-\beta) = y$, $y \in [0;1]$. Thus, we obtain $4x^2 - 4xy + 1 = 0$, which could be considered as a quadratic equation with respect to x. It follows that $D = y^2 - 1 \geq 0$ and consequently $y \in (-\infty;-1] \cup [1;+\infty)$. We get $y = 1$ and it implies that $x = \frac{1}{2}$. Further, $\cos(\alpha-\beta) = 1$ and $\cos\gamma = \frac{1}{2}$. Therefore, $\alpha = \beta$ and $\gamma = \frac{\pi}{3}$, i.e. $\alpha = \beta = \gamma = \frac{\pi}{3}$.

Problem 12. Let α, β and γ be numbers in the interval $(-\pi;\pi)$, for which $\alpha + \beta + \gamma = \pi$. Prove that $\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$ and determine when equality holds true.

Solution: Denote $\cos\alpha + \cos\beta + \cos\gamma = a$. Then

$$a = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \cos\gamma = 2\cos\frac{\pi-\gamma}{2}\cos\frac{\alpha-\beta}{2} + \cos\gamma =$$

$$2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2} \quad \Leftrightarrow 2\sin^2\frac{\gamma}{2} - 2\cos\frac{\alpha-\beta}{2}\sin\frac{\gamma}{2} + a - 1 = 0.$$

Setting $\sin\frac{\gamma}{2} = x$ we get the quadratic equation $2x^2 - 2\cos\frac{\alpha-\beta}{2}x + a - 1 = 0$, which has one real root at least. This means that its discriminant $\cos^2\frac{\alpha-\beta}{2} - 2(a-1)$ is not negative, i.e. $2a \leq \cos^2\frac{\alpha-\beta}{2} + 2$ and by $\cos^2\frac{\alpha-\beta}{2} \leq 1$ we obtain $a \leq \frac{3}{2}$. Equality holds true when $a = \frac{3}{2}$. Thus, $\alpha = \beta = \gamma = \frac{\pi}{3}$.

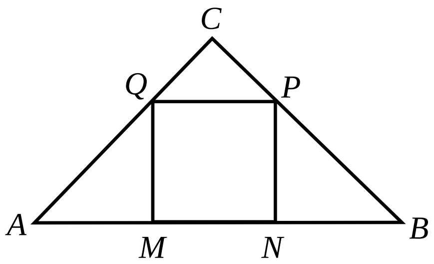
Quadratic equations, quadratic functions and their properties could be applied to geometric problems, too. This is shown by the next three problems.

Problem 13. The length of the side AC of ΔABC is equal to 6. The inner and the outer bisectors of $\angle ACB$ intersect the line AB in points L and M, respectively in such a way that A lies between L and M. If $AM = 21$ and $BL = 4$, find the length of the side BC.

Solution: By the bisector property and by $\frac{MB}{MA} = \frac{BC}{AC}$, $\frac{AL}{BL} = \frac{AC}{BC}$ it follows that $\frac{BM}{BA} = \frac{LB}{AL}$. Denote $AL = x > 0$. We get the quadratic equation $x^2 + 25x - 84 = 0$ with roots $x_1 = -\frac{56}{2}$ and $x_2 = 3$. Therefore, $AL = 3$ and the solution is $BC = 8$.

Problem 14. A square is inscribed in a rectangular triangle in such a way that two of its vertices lie on the hypotenuse, while the other two lie on the legs, respectively. The circum radius of the triangle to the square side is as $3 + 4\sqrt{3}$ to 6. Find the angles of the triangle.

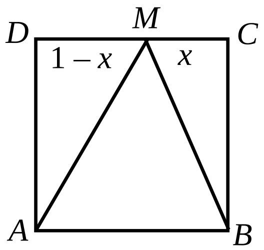
Solution: Assume that $\angle ACB = 90^\circ$ and denote the square by $MNPQ$. Let also $\angle BAC = \alpha$ and $MN = NP = PQ = QM = x$, $NB = y$, $AM = z$. From the condition, it follows that $0{,}5AB : MN = 3 + 4\sqrt{3} : 6$. From ΔAMQ we find $z = x.\cot g\alpha$ and from ΔBNP $-y = x.tg\alpha$, respectively. Since $z + x + y = AB$, we substitute and the result is $tg\alpha + \cot g\alpha = \frac{4\sqrt{3}}{3}$. If $tg\alpha = u$, then the following quadratic equation is obtained $3u^2 - 4\sqrt{3}y + 3 = 0$. The roots are $y_1 = \sqrt{3}$ and $y_2 = \frac{\sqrt{3}}{3}$. Consequently, the angles of the triangle are 90°, 60° and 30°.



Problem 15. Given is a square $ABCD$ with side 1 and a point M on the side CD.

a) Prove that $\cot g\varphi = x^2 - x + 1$, where $x = CM$ and $\varphi = \angle AMB$.

b) Find the positions of M on CD, when the values of $\angle AMB$ are the smallest and the greatest, respectively.



Solution: We restrict ourselves only to the solution of b) taking into account that from a) it follows that $0 \le x \le 1$. Since $\cot g\varphi$ is a decreasing function, we reformulate the task by it. One of the properties of the quadratic function $f(x) = x^2 - x + 1$ is $\frac{3}{4} \le f(x)$ when $0 \le x \le 1$. Its smallest value $\frac{3}{4}$ is obtained when $x = \frac{1}{2}$, while its greatest value 1 is obtained when $x = 0$ and $x = 1$. Consequently, when M coincides with C or D, the angle $\angle AMB$ is the smallest possible and it is equal to 45°. When M is the midpoint of the side CD, then $\angle AMB$ is the greatest possible.

The next problem proposes a combination of geometry, inequalities and properties of the quadratic function.

Problem 16. Prove that the positive numbers a, b and c are the lengths of a triangle sides iff the inequality $a^2 p + b\ \ q > c^2 pq$ is verified for all pairs of real numbers p and q, such that $p + q = 1$.

Solution: A triangle exists iff $a + b - c > 0$, $b + c - a > 0$ and $c + a - b > 0$. We will show that these conditions are equivalent in our case to the inequality $a^2 p + b\ \ q > c^2 pq$. Denote $K = a^2 p + b\ \ q - c^2 pq$. Then,

$$K = c^2 p^2 + (a^2 - b^2 - c^2)p + b^2 .$$

The inequality $K > 0$ is verified for all real p iff its discriminant is negative, i.e. iff $(a^2 - b^2 - c^2)^2 - 4b^2c^2 < 0$. The last is equivalent to

$$(a + b + c)(b + c - a)(c + a - b)(a + b - c) > 0 ,$$

which follows by the triangle inequality. The reverse is also true because the numbers a, b and c are positive.

The last problem is an Olympiad one which was given in Germany in 1980. It shows an application of the quadratic function properties to rational and irrational numbers.

Problem 17. Show that if a and b are positive integers and none of them is a perfect cube, then the number $\sqrt[3]{a}+\sqrt[3]{b}$ is irrational.

Solution: What will be used is the fact that if n is a positive integer and the number $\sqrt[3]{n}$ is rational, then n is a perfect cube, i.e. $\sqrt[3]{n}$ is a positive integer, too. Thus, it is enough to show that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are irrational. Assume that $\sqrt[3]{a}+\sqrt[3]{b}$ is rational and consider the quadratic equation $x^2-px+q=0$ with roots $\sqrt[3]{a}$ and $\sqrt[3]{b}$. From Viete's formulae it follows that $q=\sqrt[3]{a}.3\sqrt{b}$ and that $p=\sqrt[3]{a}+\sqrt[3]{b}$ is rational. Since $p^3=a+b+3p\sqrt[3]{a}.\sqrt[3]{b}$, it follows that q is rational, too. Let $\alpha=\sqrt[3]{a}$ and $\alpha=\sqrt[3]{b}$. Then $\alpha^2-p\alpha+q=0$. Multiplying by α, we obtain $\alpha^3-p\alpha^2+q\alpha=0$, which is equivalent to $\alpha^3-\alpha(p^2-q)+pq=0$. Since α^3, pq and $p^2-q\neq 0$ are rational, it follows that α is rational, too, i.e. $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational, contradiction. Consequently, $\sqrt[3]{a}+\sqrt[3]{b}$ is irrational.

A FINAL REMARK

The proposed problem set in the present paper could be expanded not only in the considred directions but also to applications of quadratic functions and their properties to exponential, logarithmic, trigonometric functions and other domains. The topic is rich in ideas and the authors have the intention to discus some of them on other occasions.

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