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ALMOST CONTACT B-METRIC MANIFOLDS WITH CURVATURE TENSORS OF KÄHLER TYPE

Mancho Manev, Miroslava Ivanova

Abstract. On 5-dimensional almost contact B-metric manifolds, the form of any φ-Kähler-type tensor (i.e. a tensor satisfying the properties of the curvature tensor of the Levi-Civita connection in the special class of the parallel structures on the manifold) is determined. The associated 1-forms are derived by the scalar curvatures of the φ-Kähler-type tensor for the φ-canonical connection on the manifolds from the main classes with closed associated 1-forms.

Key words: Almost contact manifold, B-metric, natural connection, canonical connection, Kähler-type tensor, totally real 2-plane, sectional curvature, scalar curvature.

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Introduction

The curvature properties of the almost contact B-metric manifolds are investigated with respect to the Levi-Civita connection ∇ and another linear connection preserving the structures of the manifold. Such connections, which curvature tensors possess the properties of the curvature tensor of ∇ in the class with ∇-parallel structures, play a significant role.

The present paper is organized as follows. In Sec. 1, we give some necessary facts about the considered manifolds. Sec. 2 is devoted to the φ-Kähler-type tensors, i.e. the tensors satisfying the properties of the curvature tensor of ∇ in the special class $\mathcal{F}_0$. In Sec. 3, it is determined the form of any φ-Kähler-type tensor L on a 5-dimensional manifold under consideration. In Sec. 4, it is proved that the associated 1-forms θ and θ^* are derived by the non-φ-holomorphic pair of scalar curvatures of the φ-Kähler-type tensor for the φ-canonical connection on the manifolds from the main classes with closed 1-forms. In Sec. 5, some of the obtained results are illustrated by a known example.

1. Preliminaries

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ,ξ,η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ, its dual 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n,n+1)$, such that the following relations are satisfied

$$\varphi\xi=0,\quad \varphi^2=-\operatorname{Id}+\eta\otimes\xi,\quad \eta\circ\varphi=0,\quad \eta(\xi)=1,$$
$$g(x,y)=-g(\varphi x,\varphi y)+\eta(x)\eta(y)$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M.

Further, x, y, z will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M is defined by

$$\tilde{g}(x,y)=g(x,\varphi y)+\eta(x)\eta(y).$$

Both metrics g and $\tilde{g}$ are necessarily of signature $(n,n+1)$. The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is also an almost contact B-metric manifold.

The structural tensor F of type (0,3) on (M,φ,ξ,η,g) is defined by the equality $F(x,y,z)=g\big((\nabla_x\varphi)\,y,z\big)$. It has the following properties:

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi).$$

The following 1-forms are associated with F:

$$\theta(z)=g^{ij}F(e_i,e_j,z),\quad \theta^*(z)=g^{ij}F(e_i,\varphi e_j,z),\quad \omega(z)=F(\xi,\xi,z),$$

where g^{ij} are the components of the inverse matrix of g with respect to a basis $\{e_i;\xi\}$ $(i=1,\,2,\,\dots\,,\,2n)$ of the tangent space T_pM of M at an arbitrary point $p\in M$. Obviously, the equality $\omega(\xi)=0$ and the relation $\theta^*\circ\varphi=-\theta\circ\varphi^2$ are always valid.

A classification of the almost contact manifolds with B-metric with respect to F is given in [3]. This classification includes eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\dots$, $\mathcal{F}_{11}$. Their intersection is the special class $\mathcal{F}_0$ determined by $F(x,y,z)=0$.

Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0$.

In the present paper we consider the manifolds from the so-called main classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_{11}$, shortly the $\mathcal{F}_i$-manifolds ($i = 1, 4, 5, 11$). These classes are the only classes where the tensor F is expressed by the metrics g and $\tilde{g}$. They are defined as follows:

$$\mathcal{F}_1: \quad F(x,y,z) = \frac{1}{2n}\big\{g(x,\varphi y)\theta(\varphi z) + g(\varphi x,\varphi y)\theta(\varphi^2 z)\big\}_{(y\leftrightarrow z)};$$

$$\mathcal{F}_4: \quad F(x,y,z) = -\frac{\theta(\xi)}{2n}\big\{g(\varphi x,\varphi y)\eta(z) + g(\varphi x,\varphi z)\eta(y)\big\};$$

$$\mathcal{F}_5: \quad F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\big\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\big\};$$

$$\mathcal{F}_{11}: \quad F(x,y,z) = \eta(x)\left\{\eta(y)\omega(z) + \eta(z)\omega(y)\right\},$$

where (for the sake of brevity) we use the denotation $\{A(x,y,z)\}_{(y\leftrightarrow z)}$ instead of $\{A(x,y,z) + A(x,z,y)\}$ for any tensor $A(x,y,z)$.

Let us remark that the class $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$ is the odd-dimensional analogue of the class $\mathcal{W}_1$ of the conformal Kähler manifolds of the almost complex manifold with Norden metric, introduced in [4].

2. Curvature-like tensors

Let $R = \big[\nabla,\nabla\big] - \nabla_{[\ ,\ ]}$ be the curvature (1,3)-tensor of the Levi-Civita connection ∇.

We denote the curvature $(0,4)$-tensor by the same letter: $R(x,y,z,w) = g(R(x,y)z,w)$.

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities are defined respectively by

$$\begin{aligned} &\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad &&\tau = g^{ij}\rho(e_i,e_j),\\ &\rho^*(y,z) = g^{ij}R(e_i,y,z,\varphi e_j), \qquad &&\tau^* = g^{ij}\rho^*(e_i,e_j). \end{aligned} \tag{2.1}$$

Definition 2.1. *([12]) Each (0,4)-tensor L on (M,φ,ξ,η,g) having the following properties is called a* curvature-like tensor*:*

$$L(x,y,z,w) = -L(y,x,z,w) = -L(x,y,w,z), \tag{2.2}$$

$$\underset{x,y,z}{\mathfrak{S}} L(x,y,z,w) = 0. \tag{2.3}$$

The above properties are a characteristic of the curvature tensor R.

Similarly to (2.1), the Ricci tensor, the scalar curvature and their associated quantities are determined for each curvature-like tensor L.

Definition 2.2. *([12]) A curvature-like tensor L on (M,φ,ξ,η,g) is called a* φ-Kähler-type tensor *if it satisfies the condition*

$$L(x,y,\varphi z,\varphi w) = -L(x,y,z,w). \tag{2.4}$$

This property is a characteristic of R on a $\mathcal{F}_0$-manifold. Moreover, (2.4) is similar to the property for a Kähler-type tensor with respect to J on an almost complex manifold with Norden metric ([1]).

Lemma 2.1. *If L is a φ-Kähler-type tensor on (M,φ,ξ,η,g), then the following properties are valid:*

$$L(\varphi x,\varphi y,z,w) = L(x,\varphi y,\varphi z,w) = -L(x,y,z,w), \tag{2.5}$$

$$L(\xi,y,z,w) = L(x,\xi,z,w) = L(x,y,\xi,w) = L(x,y,z,\xi) = 0, \tag{2.6}$$

$$L(\varphi x,y,z,w) = L(x,\varphi y,z,w) = L(x,y,\varphi z,w) = L(x,y,z,\varphi w). \tag{2.7}$$

Proof. Equalities (2.5) and (2.6) follow immediately from (2.2), (2.3) and (2.4). Properties (2.5) and (2.6) imply (2.7).

□

We consider an associated tensor L^* of L by the equality

$$L^*(x,y,z,w) = L(x,y,z,\varphi w).$$

Let us remark, the tensor L^* is not a curvature-like tensor at all. If L is a φ-Kähler-type tensor, then L^* is also a φ-Kähler-type tensor. Then the properties in Lemma 2.1 are valid for L^*. Obviously, the associated tensor of L^*, i.e. $(L^*)^*$, is $-L$. Consequently, we have the following

Corollary 2.1. *Let L and its associated tensor L^* be φ-Kähler-type tensors on (M,φ,ξ,η,g). Then we have*

$$\begin{aligned} \rho(L^*) &= \rho^*(L), \\ \rho^*(L^*) &= -\rho(L), \\ \tau(L^*) &= \tau^*(L), \\ \tau^*(L^*) &= -\tau(L). \end{aligned}$$

2.1. Examples of curvature-like tensors of φ-Kähler type

Let us consider the following basic tensors of type (0,4) derived by the structural tensors of (M,φ,ξ,η,g) and an arbitrary tensor S of type (0,2):

$$\psi_1(S)(x,y,z,w) = \big\{g(y,z)S(x,w) + g(x,w)S(y,z)\big\}_{[x\leftrightarrow y]},$$

$$\psi_2(S)(x,y,z,w) = \big\{g(y,\varphi z)S(x,\varphi w) + g(x,\varphi w)S(y,\varphi z)\big\}_{[x\leftrightarrow y]},$$

$$\begin{aligned}\psi_3(S)(x,y,z,w) = -\big\{&g(y,z)S(x,\varphi w) + g(y,\varphi z)S(x,w)\\ &+ g(x,\varphi w)S(y,z) + g(x,w)S(y,\varphi z)\big\}_{[x\leftrightarrow y]},\end{aligned}$$

$$\psi_4(S)(x,y,z,w) = \big\{\eta(y)\eta(z)S(x,w) + \eta(x)\eta(w)S(y,z)\big\}_{[x\leftrightarrow y]},$$

$$\psi_5(S)(x,y,z,w) = \big\{\eta(y)\eta(z)S(x,\varphi w) + \eta(x)\eta(w)S(y,\varphi z)\big\}_{[x\leftrightarrow y]},$$

where we use the following denotation $\{A(x,y,z)\}_{[x\leftrightarrow y]}$ instead of the difference $A(x,y,z)-A(y,x,z)$ for any tensor $A(x,y,z)$. The tensor $\psi_1(S)$ coincides with the known Kulkarni-Nomizu product of the tensors g and S.

The five tensors $\psi_i(S)$ are not curvature-like tensors at all. In [12] and [9], it is proved that on an almost contact B-metric manifold:

1. $\psi_1(S)$ and $\psi_4(S)$ are curvature-like tensors if and only if $S(x,y) = S(y,x)$;
2. $\psi_2(S)$ and $\psi_5(S)$ are curvature-like tensors if and only if $S(x,\varphi y) = S(y,\varphi x)$;
3. $\psi_3(S)$ is a curvature-like tensor if and only if $S(x,y) = S(y,x)$ and $S(x,\varphi y) = S(y,\varphi x)$.

Moreover, both of the tensors $\psi_1(S) - \psi_2(S) - \psi_4(S)$ and $\psi_3(S) + \psi_5(S)$ are of φ-Kähler type if and only if the tensor S is symmetric and hybrid with respect φ, i.e. $S(x,y) = S(y,x)$ and $S(x,y) = -S(\varphi x,\varphi y)$. In this case, their associated tensors are the following:

$$(\psi_1 - \psi_2 - \psi_4)^*(S) = -(\psi_3 + \psi_5)(S),$$
$$(\psi_3 + \psi_5)^*(S) = (\psi_1 - \psi_2 - \psi_4)(S).$$

The following tensors π_i $(i = 1, 2, \dots, 5)$, derived only by the metric tensors of (M,φ,ξ,η,g), play an important role in differential geometry of an almost contact B-metric manifold:

$$\pi_i = \frac{1}{2}\psi_i(g),\ (i = 1, 2, 3); \qquad \pi_i = \psi_i(g),\ (i = 4, 5).$$

In [12], it is proved that π_i $(i = 1, 2, \dots, 5)$ are curvature-like tensors and the tensors

$$L_1 = \pi_1 - \pi_2 - \pi_4, \qquad L_2 = \pi_3 + \pi_5$$

are φ-Kähler-type tensors. Their associated φ-Kähler-type tensors are as follows

$$L_1^* = -L_2, \qquad L_2^* = L_1.$$

3. φ-Kähler-type tensors on a 5-dimensional almost contact B-metric manifold

Let α be a non-degenerate totally real section in T_pM, $p \in M$, and α be orthogonal to ξ with respect to g, i.e. $\alpha \perp \varphi\alpha$, $\alpha \perp \xi$. Let $k(\alpha;p)(L)$ and $k^*(\alpha;p)(L)$ be the scalar curvatures of α with respect to a curvature-like tensor L, i.e.

$$k(\alpha;p)(L) = \frac{L(x,y,y,x)}{\pi_1(x,y,y,x)}, \qquad k^*(\alpha;p)(L) = \frac{L(x,y,y,\varphi x)}{\pi_1(x,y,y,x)},$$

where $\{x,y\}$ is an arbitrary basis of α.

We recall two known propositions for constant sectional curvatures.

Theorem 3.1. *([16]) Let (M,φ,ξ,η,g) $(\dim M \geq 5)$ be an almost contact B-metric $\mathcal{F}_0$-manifold. Then M is of constant totally real sectional curvatures $\nu = \nu(p)(R) = k(\alpha;p)(R)$ and $\nu^* = \nu^*(p)(R) = k^*(\alpha;p)(R)$ if and only if $R = \nu L_1 + \nu^* L_2$. Both functions ν and ν^* are constant if M is connected and* $\dim M \geq 7$.

Theorem 3.2. *([17]) Each 5-dimensional almost contact B-metric $\mathcal{F}_0$-manifold has point-wise constant totally real sectional curvatures*

$$\nu(p)(R) = k(\alpha;p)(R), \qquad \nu^*(p)(R) = k^*(\alpha;p)(R).$$

In this relation, we give the following

Theorem 3.3. *Let (M,φ,ξ,η,g) be a 5-dimensional almost contact B-metric manifold. Then each φ-Kähler-type tensor has the form*

$$L = \nu L_1 + \nu^* L_2,$$

where $\nu = \nu(L)$ and $\nu^* = \nu^*(L) = \nu(L^*)$ are the sectional curvatures of the totally real 2-planes orthogonal to ξ in T_pM, $p \in M$, with respect to L. Moreover, (M,φ,ξ,η,g) is of point-wise contact sectional curvatures of the totally real 2-planes orthogonal to ξ with respect to L.

Proof. Let $\{e_1, e_2, \varphi e_1, \varphi e_2, \xi\}$ be an adapted φ-basis of T_pM with respect to g, i.e.

$$-g(e_1,e_1) = -g(e_2,e_2) = g(\varphi e_1,\varphi e_1) = g(\varphi e_2,\varphi e_2) = 1,$$
$$g(e_i,\varphi e_j) = 0, \quad \eta(e_i) = 0 \quad (i,\, j \in \{1,2\}).$$

Then an arbitrary vector in T_pM has the form $x = x^1 e_1 + x^2 e_2 + \tilde{x}^1 \varphi e_1 + \tilde{x}^2 \varphi e_2 + \eta(x)\xi$. Using properties (2.2), (2.3) and (2.4) for $L(x,y,z,w)$, we obtain immediately $L = \nu L_1 + \nu^* L_2$, where $\nu = L(e_1,e_2,e_2,e_1)$, $\nu^* = L(e_1,e_2,e_2,\varphi e_1) = \nu(L^*) = L^*(e_1,e_2,e_2,e_1)$ are the sectional curvatures of α with respect to L, because $\pi_1(e_1,e_2,e_2,e_1) = 1$.

Then, if $\{x,y\}$ is an adapted φ-basis of an arbitrary totally real 2-plane α orthogonal to ξ, i.e.

$$g(x,y) = g(x,\varphi x) = g(x,\varphi y) = g(y,\varphi y) = \eta(x) = \eta(y) = 0,$$

we get $k(\alpha;p)(L) = \nu(p)(L)$, $k^*(\alpha;p)(L) = \nu^*(p)(L)$, taking into account the expression $L = \nu L_1 + \nu^* L_2$. Therefore, (M,φ,ξ,η,g) is of point-wise contact sectional curvatures of α with respect to L.

□

The restriction of Theorem 3.3 to $\mathcal{F}_0$ coincides with Theorem 3.1 because R is a φ-Kähler-type tensor on a $\mathcal{F}_0$-manifold.

3.1. Curvature tensor of a natural connection on a 5-dimensional almost contact B-metric manifold

In [10], it is introduced the notion of a *natural connection* on the manifold $(M,\, \varphi,\, \xi,\, \eta,\, g)$ as a linear connection D, with respect to which the almost contact structure (φ,ξ,η) and the B-metric g are parallel, i.e. $D\varphi = D\xi = D\eta = Dg = 0$. According to [13], a necessary and sufficient condition a linear connection D to be natural on (M,φ,ξ,η,g) is $D\varphi = Dg = 0$.

Let K be curvature tensor of a natural connection D with torsion T. Then K satisfies (2.2) and (2.4). Instead of (2.3), we have the following form of the first Bianchi identity ([5])

$$\underset{x,y,z}{\mathfrak{S}} K(x,y,z,w) = \underset{x,y,z}{\mathfrak{S}} \left\{ T(T(x,y),z,w) + (D_x T)(y,z,w) \right\}.$$

If we set the condition $\underset{x,y,z}{\mathfrak{S}} K(x,y,z,w) = 0$ as for the curvature tensor R, then K is a φ-Kähler-type tensor and satisfies the condition of Theorem 3.3. Therefore, we obtain

Corollary 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a 5-dimensional almost contact B-metric manifold with a natural connection D with curvature tensor K of φ-Kähler-type. Then K has the form*

$$K = \nu L_1 + \nu^* L_2,$$

where $\nu = \nu(K)$ and $\nu^* = \nu^*(K) = \nu(K^*)$ are the sectional curvatures of the totally real 2-planes orthogonal to ξ in T_pM, $p \in M$, with respect to K. Moreover, $(M, \varphi, \xi, \eta, g)$ is of point-wise contact sectional curvatures of the totally real 2-planes orthogonal to ξ with respect to K.

4. Curvature tensor of the φ-canonical connection

According to [15], a natural connection D is called a *φ-canonical connection* on the manifold $(M, \varphi, \xi, \eta, g)$ if the torsion tensor T of D satisfies the following identity:

$$\begin{aligned}&\big\{T(x,y,z) - T(x,\varphi y,\varphi z) - \eta(x)\{T(\xi,y,z) - T(\xi,\varphi y,\varphi z)\}\\&- \eta(y)\{T(x,\xi,z) - T(x,z,\xi) - \eta(x)T(z,\xi,\xi)\}\big\}_{[y\leftrightarrow z]} = 0.\end{aligned}$$

Let us remark that the restriction the φ-canonical connection D of the manifold $(M, \varphi, \xi, \eta, g)$ on the contact distribution $\ker(\eta)$ is the unique canonical connection of the corresponding almost complex manifold with Norden metric, studied in [2].

In [12], it is introduced a natural connection on $(M, \varphi, \xi, \eta, g)$, defined by

$$D_x y = \nabla_x y + \frac{1}{2}\big\{(\nabla_x \varphi)\,\varphi y + (\nabla_x \eta)\, y \cdot \xi\big\} - \eta(y)\nabla_x \xi. \tag{4.1}$$

In [14], the connection determined by (4.1) is called a *φB-connection.* It is studied for some classes of $(M, \varphi, \xi, \eta, g)$ in [12], [6], [7] and [14]. The φB-connection is the odd-dimensional counterpart of the B-connection on the corresponding almost complex manifold with Norden metric, studied for the class $\mathcal{W}_1$ in [1].

In [15], it is proved that the φ-canonical connection and the φB-connection coincide on the almost contact B-metric manifolds in a class which contains $\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{11}$.

According to [12], the necessary and sufficient conditions K to be a φ-Kähler-type tensor in $\mathcal{F}_i$ ($i = 1, 4, 5, 11$) is the associated 1-forms θ, θ^* and $\omega \circ \varphi$ to be closed. These subclasses we denote by $\mathcal{F}_i^0$ ($i = 1, 4, 5, 11$).

Bearing in mind the second Bianchi identity

$$\underset{x,y,z}{\mathfrak{S}}\left\{\left(D_xK\right)(y,z)+K\left(T(x,y),z\right)\right\}=0,$$

we compute the scalar curvatures for K determined by

$$\tau(K)=g^{ij}\rho(K)_{ij},\qquad \tau^*(K)=\tau(K^*)=g^{ij}\varphi_j^k\rho(K)_{ik},$$

where $\rho(K)_{ij}$ is the Ricci tensor of K, and then we get the following

Lemma 4.2. *For* (M,φ,ξ,η,g) *in* $\mathcal{F}_i^0$ $(i=1,4,5,11)$, *the relations for the scalar curvatures* $\tau=\tau(K)$ *and* $\tau^*=\tau^*(K)$ *of* K *are:*

$$\text{(4.2)}\qquad \mathrm{d}\tau\circ\varphi=-\mathrm{d}\tau^*-\frac{1}{n}\left(\tau\theta+\tau^*\theta^*\right),\quad \mathrm{d}\tau^*\circ\varphi=\mathrm{d}\tau-\frac{1}{n}\left(\tau^*\theta-\tau\theta^*\right).$$

Obviously, bearing in mind (4.2), we it follows that the pair (τ,τ^*) on (M,φ,ξ,η,g) is a φ-holomorphic pair of functions, i.e. $\mathrm{d}\tau=\mathrm{d}\tau^*\circ\varphi$ and $\mathrm{d}\tau^*=-\mathrm{d}\tau\circ\varphi$, if and only if the associated 1-forms θ and θ^* are zero. Such one is the case for the class $\mathcal{F}_{11}$.

The system (4.2) can be solved with respect to θ and θ^* and then

$$\text{(4.3)}\qquad \theta=-n\left\{\mathrm{d}f_1+\mathrm{d}f_2\circ\varphi\right\},\qquad \theta^*=n\left\{\mathrm{d}f_1\circ\varphi-\mathrm{d}f_2\right\},$$

where $f_1=\arctan\left(\tau^*/\tau\right)$, $f_2=\ln\sqrt{\tau^2+\tau^{*2}}$.

Let us consider the complex-valued function $h=\tau+i\tau^*$ or in polar form $h=|h|e^{i\alpha}$. Then we have $|h|=\sqrt{\tau^2+\tau^{*2}}$, $\alpha=\arctan\left(\tau^*/\tau\right)$.

Bearing in mind that $\operatorname{Log}h=\ln|h|+i\alpha$, then (4.3) take the following form:

$$\text{(4.4)}\qquad \theta=-n\left\{\mathrm{d}\alpha+\mathrm{d}(\ln|h|)\circ\varphi\right\},\quad \theta^*=n\left\{\mathrm{d}\alpha\circ\varphi-\mathrm{d}(\ln|h|)\right\}.$$

So, we obtain the following

Theorem 4.4. *For* (M,φ,ξ,η,g) *in* $\mathcal{F}_i^0$ $(i=1,4,5)$, *the associated 1-forms* θ *and* θ^* *are derived by the non-*φ*-holomorphic pair of the scalar curvatures* (τ,τ^*) *of the* φ*-Kähler-type tensor* K *for the* φ*-canonical connection* D *by* (4.4).

Corollary 4.3.
For $i=1$

$$\theta=n\left\{\mathrm{d}\alpha\circ\varphi^2-\mathrm{d}(\ln|h|)\circ\varphi\right\},\qquad \theta^*=n\left\{\mathrm{d}\alpha\circ\varphi+\mathrm{d}(\ln|h|)\circ\varphi^2\right\};$$

For $i=4$

$$\theta=-n\mathrm{d}\alpha(\xi)\eta,\qquad \theta^*=0;$$

For $i = 5$
$$\theta = 0, \qquad \theta^* = -n\mathrm{d}(\ln|h|)(\xi)\eta.$$

5. Examples of almost contact manifolds with B-metric

Let us consider $\mathbb{R}^{2n+2} = \left\{\left(u^1, \dots, u^{n+1}; v^1, \dots, v^{n+1}\right) \mid u^i, v^i \in \mathbb{R}\right\}$ as a complex Riemannian manifold with the canonical complex structure J and a metric g, defined by $g(x,x) = -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j$, where $x = \lambda^i\frac{\partial}{\partial u^i} + \mu^i\frac{\partial}{\partial v^i}$. Identifying the point $p = \left(u^1, \dots, u^{n+1}; v^1, \dots, v^{n+1}\right)$ in $\mathbb{R}^{2n+2}$ with its positional vector Z, in [3] it is given a hypersurface S defined by

$$g(Z, JZ) = 0, \quad g(Z,Z) = \cosh^2 t, \quad t > 0.$$

The almost contact structure is determined by the conditions:

$$\xi = \frac{1}{\cosh t}Z, \qquad Jx = \varphi x + \eta(x)J\xi,$$

where $x,\ \varphi x \in T_pS$ and $J\xi \in (T_pS)^\perp$. Then $(S, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold in the class $\mathcal{F}_5$.

Consequently, we characterize $(S, \varphi, \xi, \eta, g)$ by means of [8]. We compute the following quantities for the constructed $\mathcal{F}_5$-manifold:

$$\theta = 0, \qquad \eta = \sinh t\mathrm{d}t, \qquad \frac{\xi\theta^*(\xi)}{2n} = -\frac{\theta^{*2}(\xi)}{4n^2} = -\frac{1}{\cosh^2 t}. \tag{5.1}$$

In [11], it is given that the 1-form θ^* on a $\mathcal{F}_5$-manifold is closed if and only if $x\theta^*(\xi) = \xi\theta^*(\xi)\eta(x)$. By virtue of (5.1), we establish that $(S, \varphi, \xi, \eta, g)$ belongs to the subclass $\mathcal{F}_5^0$, since $\mathrm{d}\theta^* = 0$.

The condition for the second fundamental form of the hypersurface S, given in [3], the Gauss equation ([5]) and the flatness of $\mathbb{R}^{2n+2}$ imply the following form of the curvature tensor of ∇

$$R = -\frac{1}{\cosh^2 t}\pi_2.$$

Then, taking into account (5.1) and the form of the curvature tensor K of the φ-canonical connection in $\mathcal{F}_5^0$ ([12])

$$K = R + \frac{\xi\theta^*(\xi)}{2n}\pi_4 + \frac{\theta^{*2}(\xi)}{4n^2}\pi_1,$$

we obtain

$$K = \frac{1}{\cosh^2 t} L_1. \tag{5.2}$$

Since L_1 is a φ-Kähler-type tensor, then K is also a φ-Kähler-type tensor. Therefore, we have

$$\nu(K) = K(e_1, e_2, e_2, e_1) = \frac{1}{\cosh^2 t}, \qquad \nu^*(K) = K^*(e_1, e_2, e_2, e_1) = 0,$$

which illustrates Theorem 3.3 and Corollary 3.2.

According to (5.2), the scalar curvatures are

$$\tau(K) = \frac{4n(n-1)}{\cosh^2 t}, \qquad \tau^*(K) = 0.$$

Then, taking into account (5.1), the results for $(S, \varphi, \xi, \eta, g)$ illustrate also Lemma 4.2, Theorem 4.4 and Corollary 4.3.

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M. Manev, M. Ivanova,
Department of Algebra and Geometry
Faculty of Mathematics and Informatics
University of Plovdiv,
236 Bulgaria Blvd
4000 Plovdiv, Bulgaria
e-mail: mmanev@uni-plovdiv.bg
mirkaiv@uni-plovdiv.bg

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ПОЧТИ КОНТАКТНИ B-МЕТРИЧНИ МНОГООБРАЗИЯ С КРИВИННИ ТЕНЗОРИ ОТ КЕЛЕРОВ ТИП

Манчо Манев, Мирослава Иванова

Резюме. Определен е видът на всеки тензор от φ-келеров тип (т.е. тензор, удовлетворяващ свойствата на тензора на кривина за свързаността на Леви-Чивита в специалния клас на паралелните структури върху многообразието) върху 5-мерни почти контактни B-метрични многообразия. Асоциираните 1-форми се пораждат от скаларните кривини на тензора от φ-келеров тип за φ-каноничната свързаност върху многообразията от главните класове със затворени асоциирани 1-форми.