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# METHOD OF PERTURBING FAMILIES OF LYAPUNOV FUNCTIONS FOR INVESTIGATION OF THE STABILITY IN TERMS OF TWO MEASURES

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**Abstract**

The stability in terms of two measures is studied by the help with the method of perturbing families of Lyapunov functions.

## 1. Introduction

It has been demonstrated [3], [5] that using technique of perturbing Lyapunov functions and employing a family of Lyapunov functions are helpful in discussing nonuniform properties of solutions of differential systems under weaker assumptions.

In [1], the authors discuss nonuniform stability properties in terms of two measures employing perturbing families of Lyapunov functions.

Lakshmikantham V. and his followers fully develop [2] the method of vector Lyapunov functions by combining the ideas involved in the foregoing techniques and this helps in distributing the burden between groups of components of the vector Lyapunov functions and the comparison systems.

## 2. Preliminary results

We consider the initial value problem for the system of differential equations

$$\dot{x} = f(t, x) \tag{1}$$

$x(t_0) = x_0$, where $x \in R^n$, $f \in C[R^+ \times R^n, R^n]$ and $f(t, 0) \equiv 0$.

We will assume that there exists a solution $x(t)$, $t \geq t_0$ of the initial value problem (1) for every point $(t_0, x_0) \in R^+ \times R^n$.

We consider the initial value problem of the following comparison system

$$\dot{u} = g(t, u) \tag{2}$$

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$u(t_0) = u_0 \geq 0$, where $u \in R^N$, $N \leq n$, $g \in C[R^+ \times R^N, R^N]$ and $g(t, 0) \equiv 0$.

Let $p$ and $q$ are fixed natural numbers such that $p + q = N$. We deduce the following notation

$$u = (u_p, u_q) = (u_1, u_2, \ldots, u_p, u_{p+1}, \ldots, u_N).$$

According to the notation mentioned above, the group of components $u_p$ of a vector $u \in R^N$ contain the first $p$ element of $u$, and the group of components $u_q$ — the last $N - p = q$ elements of a vector $u$. We note that not regarding the restriction, we can assume that $u_p$ contain any $p$ elements of a vector $u$, and $u_q$ — the rest $N - p = q$ elements of a vector $u$.

We will define the following classes of functions:

$$\begin{aligned} K^* &= [\sigma \in C[R^+, R^+] : \sigma(u) \text{ is strictly increasing and } \sigma(0) = 0] \\ CK^* &= [\sigma \in C[R^+ \times R^+, R^+] : \sigma(t, u) \in K^* \text{ for each } t \in R^+] \\ \Gamma &= [h \in C[R^+ \times R^n, R^+] : \inf_{x \in R^n} h(t, x) = 0 \text{ for each } t \in R^+] \end{aligned}$$

**Definition 1** [1]. Let $h_0, h \in \Gamma$. Then we say that $h_0$ is finer that $h$ if there exist a number $\rho > 0$ and a function $\Phi \in K^*$ such that $h_0(t, x) < \rho$ implies $h(t, x) \leq \Phi(h_0(t, x))$.

**Definition 2** [1]. The system (1) is said to be $(h_0, h)$-equistable, if given $\varepsilon > 0$ and $t_0 \in R^+$ there exists a $\delta = \delta(t_0, \varepsilon)$ that is continuous in $t_0$ for each $\varepsilon$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0$.

**Definition 3** [2]. Let $Q \in C[R^N_+, R^+]$ with $Q(0) = 0$ and $Q(u)$ is nondecreasing in $u$. Then we say that $Q \in K[R^N_+, R^+]$.

**Definition 4** [2]. Let $V \in C[R^+ \times R^n, R^N]$, $h_0$, $h \in \Gamma$ and a function $Q \in K[R^N_+, R^+]$. Then $V$ is said to be:

1) $h$-positive definite if there exist a number $\rho > 0$ and a function $b \in K^*$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq Q(V(t, x))$;

2) $h_0$-decrescent if there exist a number $\rho_0 > 0$ and a function $a_0 \in K^*$ such that $h_0(t, x) < \rho_0$ implies $Q(V(t, x)) \leq a_0(h_0(t, x))$;

3) weakly $h_0$-decrescent if there exist a number $\rho_0 > 0$ and a function $a \in CK^*$ such that $h_0(t, x) < \rho_0$ implies $Q(V(t, x)) \leq a(t, h_0(t, x))$.

**Definition 5** [2]. Let $Q_1 \in K[R^p_+, R^+]$, $Q_2 \in K[R^q_+, R^+]$ and $u(t; t_0, u_0)$ be any solution of the system (2) existing for all $t \geq t_0$. Then the zero solution of the system (2) is said to be equi-uniform stable if for given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $t_0 \in R^+$ there exist $\delta_1 = \delta_1(t_0, \varepsilon_1) > 0$, $\delta_2 = \delta_2(\varepsilon_2)$ such that

$$Q_1(u_{0p}) < \delta_1 \text{ implies } Q_1(u_p(t; t_0, u_0)) < \varepsilon_1, \quad t \geq t_0$$

and

$$Q_2(u_{0q}) < \delta_2 \text{ implies } Q_2(u_q(t; t_0, u_0)) < \varepsilon_2, \quad t \geq t_0.$$

We assume that the right parts of the system (2) are defined and continuous in the open domain $G \subset R^{N+1} = \{t, u_1, \dots, u_N\}$ and in this domain satisfy the Wazewski's condition.

Wazewski's condition [6]. Each of the function $g_s(t, u)$ $(s = \overline{1, N})$ is nondecreasing in $u_1, \dots, u_{s-1}, u_{s+1}, \dots, u_N$, i.e. $u_1' \le u_1'', \dots, u_{s-1}' \le u_{s-1}'', u_s' = u_s'', u_{s+1}' \le u_{s+1}'', \dots, u_N' \le u_N''$ implies $g_s(t, x') \le g_s(t, x'')$.

## 3. Main results

We will give some sufficient conditions for stability in terms of two measures.

**Theorem** *Let the following hypotheses be fulfilled:*
$(\mathrm{H}_0)$ *$h_0, h \in \Gamma$ and $h_0$ is finer than $h$;*
$(\mathrm{H}_1)$ *$V \in C[S(h, \rho), R_+^N]$, $V(t, x)$ is locally Lipschitzian in $x$,*
*$S(h, \rho) = \{(t, x)\colon t \in R^+, h(t, x) < \rho\}$, $V_p(t, x)$ is weakly $h_0$-decrescent and*

$$b(h(t, x)) \le Q_2(V_q(t, x)) \le a_0(h_0(t, x)) + a_1(Q_1(V_p(t, x)))$$

*for $(t, x) \in S(h, \rho) \cap S^c(h_0, \eta)$ for every $0 < \eta < \rho$ and $Q_1(V_p(t, 0)) \equiv 0$ where $Q_1 \in K[R_+^p, R^+]$, $Q_2 \in K[R_+^q, R^+]$ and $b, a_0, a_1 \in K^*[R^+, R^+]$ with $p + q = N$;*
$(\mathrm{H}_2)$ *Each of the functions $g_s(t, V)$ $(s = \overline{1, N})$ is nondecreasing in $V_1, \dots, V_{s-1}, V_{s+1}, \dots, V_N$ i.e. fulfils the Wazewski's condition;*
*1) $D^+ V_p(t, x) \le g_p(t, V_p(t, x), 0)$, $\quad (t, x) \in S(h, \rho)$*
*2) $D^+ V_q(t, x) \le g_q(t, V(t, x))$, $\quad (t, x) \in S(h, \rho) \cap S^c(h_0, \eta)$*
*for every $0 < \eta < \rho$, where $S^c(h_0, \eta)$ is the complement of $S(h_0, \eta)$;*
$(\mathrm{H}_3)$ *the zero solution of the system (2) is equi-uniform stable.*

*Then, the differential system (1) is $(h_0, h)$-equistable.*

*Proof:* Since $V_p(t, x)$ is weakly $h_0$-decrescent, there exists a $\rho_1$ $(0 < \rho_1 \le \rho)$ and a $\Phi_0 \in CK^*$ such that

$$Q_1(V_p(t, x)) \le \Phi_0(t, h_0(t, x)) \quad \text{if} \quad h_0(t, x) < \rho_1 \tag{3}$$

Also, $h_0$ is finer than $h$ implies that there exists a $\rho_0$ $(0 < \rho_0 \le \rho_1)$ and a $\Phi_1 \in K^*$ such that

$$h(t, x) \le \Phi_1(h_0(t, x)) \quad \text{provided} \quad h_0(t, x) < \rho_0 \tag{4}$$

where $\rho_0$ is such that $\Phi_1(\rho_0) < \rho_1$.

Let $0<\varepsilon<\rho$ and $t_0 \in R^+$ be given. By hypothesis $(\mathrm{H}_3)$ given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $t_0 \in R^+$, there exist $\delta_{10} = \delta_{10}(t_0, \varepsilon_1) > 0$ and $\delta_{20} = \delta_{20}(\varepsilon_2) > 0$ such that

$$\begin{aligned} &Q_1(u_{0p}) < \delta_{10} \ \text{implies}\ Q_1(u_p(t; t_0, u_0)) < \varepsilon_1, \quad t \ge t_0 \\ &\text{and} \\ &Q_2(u_{0q}) < \delta_{20} \ \text{implies}\ Q_2(u_q(t; t_0, u_0)) < \varepsilon_2, \quad t \ge t_0 \end{aligned} \tag{5}$$

Since $a_0$, $\Phi_1 \in K^*$, we can find a $\delta_1 = \delta_1(\varepsilon)$ such that

$$a_0(\delta_1) < \tfrac{1}{2}\delta_{20} \qquad \text{and} \qquad \Phi_1(\delta_1) < \varepsilon. \tag{6}$$

Let $\varepsilon_2 = b(\varepsilon)$ and $\varepsilon_1 = a_1^{-1}(\frac{1}{2}\delta_{20})$. Choose $u_{0p} = V_p(t_0, x_0)$. Since $\Phi_0 \in CK^*$, $Q_1(V_p(t,0)) \equiv 0$ and (3), it follows that there exists a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$, $\delta_2 < \min(\delta_1, \rho_1)$ and

$$h_0(t_0, x_0) < \delta_2 \text{ implies } Q_1(V_p(t_0, x_0)) \le \Phi_0(t_0, h_0(t_0, x_0)) < \delta_{10}. \tag{7}$$

We set $\delta = \min(\delta_1, \delta_2)$ and suppose that $h_0(t_0, x_0) < \delta$. We note that because of (4) and (6), we have

$$h(t_0, x_0) \le \Phi_1(h_0(t_0, x_0)) \le \Phi_1(\delta) \le \Phi_1(\delta_1) < \varepsilon \tag{8}$$

We claim that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon,\ t \ge t_0$. Assume the contrary, i.e. according (8), there exists a solution $x(t)$ of the system (1) with $h_0(t_0, x_0) < \delta$ and $t_2 > t_1 > t_0$ such that

$$\begin{gathered} h(t_2, x(t_2)) = \varepsilon < \rho, \quad h_0(t_1, x(t_1)) = \delta_1(\varepsilon) \\ \text{and} \\ x(t) \in S(h, \varepsilon) \cap S^c(h_0, \eta) \qquad \text{with } \eta = \delta_1(\varepsilon) \text{ for } t \in [t_1, t_2]. \end{gathered} \tag{9}$$

It then follows from ($H_2$) that

$$\begin{aligned} D^+ m_p(t) &\le g_p(t, m_p(t), 0), && t_0 \le t \le t_2 \\ D^+ m_q(t) &\le g_q(t, m(t)), && t_1 \le t \le t_2 \end{aligned} \tag{10}$$

where $m(t) = V(t, x(t))$. Hence by the comparison theorem [4] we have for $t_1 \le t \le t_2$

$$m_p(t) \le u_p(t; t_1, m(t_1)), \quad m_q(t) \le u_q(t; t_1, m(t_1)) \tag{11}$$

Let $u^*(t) = u(t; t_1, m(t_1)) \ge 0$ be the extension of $u(t)$ to the left of $t_1$ up to $t_0$ and let $u^*(t_0) = u_0^*$. Choose $u_p(t_0) = V_p(t_0, x_0)$ and $u_q(t_0) = u_{0q}^*$. Consider now the differential inequality which results from (10)

$$D^+ m_p(t) \le g_p(t, m_p(t), u_q^*(t)), \quad u_p(t_0) = m_p(t_0)$$

which by comparison theorem [4] yields

$$m_p(t) \le u_p(t; t_0, u_0), \quad t_0 \le t \le t_1, \quad u_0 = (u_p(t_0), u_{0q}^*). \tag{12}$$

Then it is clear that $u(t) = (u_p(t; t_0, u_0), u_q^*(t))$ is a solution of the system (2) on $[t_0, t_1]$. Using (9), (11) and ($H_1$), we obtain

$$b(\varepsilon) = b(h(t_2, x(t_2)) \le Q_2(V_q(t_2, x(t_2))) \le Q_2(u_q(t_2; t_1, m(t_1))) \tag{13}$$

But from (5) and (12), we get

$$Q_1(V_p(t_1, x(t_1))) \le Q_1(u_p(t_1; t_0, u_0)) \le a_1^{-1}(\tfrac{1}{2}\delta_{20}(\varepsilon))$$

provided $Q_1(u_{0p}) < \delta_{10}$. From ($H_1$), (6) and (9) we have now

$$\begin{aligned} Q_2(V_q(t_1, x(t_1))) &\le a_0(h_0(t_1, x(t_1))) + a_1(Q_1(V_p(t_1, x(t_1))) \le \\ &\le a_0(\delta_1(\varepsilon)) + a_1(a_1^{-1}(\tfrac{1}{2}\delta_{20})) < \delta_{20} \end{aligned}$$

and therefore from (5) we get

$$Q_2(u_q(t_2; t_1, m(t_1))) < b(\varepsilon)$$

which contradicts (13). Hence the proof is complete.

## References


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