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CHARACTERIZATION OF A FOUR-DIMENSIONAL GLOBALLY OSSERMAN MANIFOLDS USING TRACE AND DETERMINANT OF JACOBI OPERATOR

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Abstract

Let (M, g) be a four-dimensional Riemannian manifold. The Jacobi operator R_X is symmetric linear endomorphism of the tangent space M_p at a point $p \in M$ defined by $R_X(u) = R(u, X, X)$, where X always belongs to the unit sphere S_pM at p . If eigenvalues of R_X are pointuise constants on M, then (M, g) is called pointwise Osserman manifold. In this paper we prove that (M, g) is a four-dimensional Riemannian manifold such that trace and determinant of Jacobi operator R_X are a globally constant on M if and only if (M, g) (almost everywhere) locally is a globally Osserman manifold.

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Let (M, g) be an n-dimensional Riemannian manifold with metric tensor g and curvature tensor R. The Jacobi operator R_X is a symmetric linear endomorphism of the tangent space M_p at a point $p \in M$ defined by $R_X(u) = R(u, X, X)$, where X belongs to the unit sphere S_pM. Since X is eigenvector of R_X with the corresponding eigenvalue 0, then the characteristic equation

$$det(R(e_i, X, X, e_j) - cg_{ij}) = 0 \tag{1}$$

of R_X with a root c can be represented in another form

$$c(c^{n-1} + J_1c^{n-2} + ... + J_{n-2}c + J_{n-1}) = 0, \tag{2}$$

where $J_i = J_i(p; X)$, $(i = 1, 2, ..., n-1)$. We have from (1) that $J_1(p; X) = traceR_X$ and $J_{n-1}(p; X) = detR_X$. A Riemannian manifold (M, g) is called a globally Osserman manifold if the eigenvalues of Jacobi operator R_X are a globally constants on M, respectivelly (M, g) is called a pointwise Osserman manifold if the eigenvalues of Jacobi operator R_X are a pointwise constants on M [1] . A gobally Osserman manifolds was investigated from Quo-Shin-Chi which proved that an $n-$ dimensional Riemannian manifold (M, g) is a globally Osserman manifold if and only if (M, g) locally is a symmetric space of rank 1 or (M, g) is a space of constant sectional curvature, and it holds when $n \equiv 1(mod2)$, $n \equiv 2(mod4)$ and $n = 4$ [7]. The Osserman conjecture was generalized using characteristic coefficients of Jacobi operator in the case $dimM = 4$ as follows:

Theorem 1 *[10] A four-dimensional Riemannian manifold (M,g) (almost everywhere) locally is a globally Osserman manifold if and only if the characteristic coefficients J_1 and J_2 of Jacobi operator R_X are a globally constants on M.*

Further in the present paper we will use the following:

Proposition 1 *[3] [9] If a four-dimensional Riemannian manifold (M, g) is a pointwise Osserman manifold, then is hold transformation:*

$$\begin{array}{rcl} Y & = & \alpha X + \beta X_1 + \gamma X_2 + \delta X_3, \\ Y_1 & = & -\beta X + \alpha X_1 - \delta X_2 + \gamma X_3, \\ Y_2 & = & -\gamma X + \delta X_1 + \alpha X_2 - \beta X_3, \\ Y_3 & = & -\delta X - \gamma X_1 + \beta X_2 + \gamma X_3, \end{array} \tag{3}$$

for the eigenvectors X, X_1, X_2, X_3 of the Jacobi operator R_X and for the eigenvectors Y, Y_1, Y_2, Y_3 of the Jacobi operator R_Y , where $X, Y \in S_pM$ and hence $\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$.

Q. Sh. Chi proved either that if (M, g) is a four-dimensional pointwise Osserman manifold, then the eigenvector fields of any Jacobi operator R_X (say X, A, B, C) are a smooth vector fields suppose defined in a neighborhood U_p at a point $p \in M$.

We remark that at a single point $p \in M$ holds $X\,|_p = X, X_1\,|_p = A, X_2\,|_p = B, X_3\,|_p = C$. Stanilov and Belger proved the following:

Proposition 2 *[9] Let (M, g) be a pointwise Osserman manifold. Then at a neighborhoud of any point of manifold the eigenvalues a, b, c and the eigenvector fields X, A, B, C of any Jacobi operator R_X satisfied the following two systems:*

$$\begin{array}{c} \varphi(c-b) + \psi(c-a) = 0, \\ \psi(a-c) + \theta(a-b) = 0, \\ \theta(b-a) + \phi(b-c) = 0; \end{array} \tag{4}$$

and

$$\begin{array}{c} X(a) = (\mu+\nu)a - \nu b - \mu c, \\ X(b) = -\nu a + (\nu+\lambda)b - \lambda c, \\ X(c) = -\mu a - \lambda b + (\lambda+\mu)c, \end{array} \tag{5}$$

where

$$\begin{array}{c} \varphi = g(\nabla_A B, C), \;\; \psi = g(\nabla_B C, A), \;\; \theta = g(\nabla_C A, B), \\ \lambda = g(\nabla_A A, X), \;\; \mu = g(\nabla_B B, X), \;\; \nu = g(\nabla_C C, X). \end{array}$$

Our main aim in this note is to complete the result of Theorem 1 proving the following:

Theorem 2 *Let (M, g) be a four-dimensional Riemannian manifold. Then (M, g) (almost everywhere) locally is a globally Osserman manifold if and only if the characteristic coefficients J_1 and $J_3 \neq 0$ (trace and determinant) of Jacobi operator R_X are a globally constants on M.*

Proof. Because the only if part is trivial we will prove in the sequel only the if part. From our assumption J_1 and $J_3 \neq 0$ to be globally constants on M it follows that J_1 and $J_3 \neq 0$ are pointwise constants on M , which is hold if and only if (M, g) is a pointwise Osserman [8]. If we denote by $\alpha(p)$ the matrix of the system (5) considering a, b, c as variables, then according to our hypothesis (M, g) to be a pointwise Osserman it follows that $\alpha(p)$ has a pontwise entries and also $Rank\alpha(p)$ is a pointwise function at any point $p \in M$. All possibilities cases of $Rank\alpha(p)$ are $Rank\alpha(p) = 1, 2, 3$ In these cases any Jacobi operator R_X has respectivelly $1, 2, 3$ eigenvalues. Further we will consider all this cases separatelly.

At first we denote by Ω_i the subsets of M defined by the property that any Jacobi operator R_X has i eigenvalues on Ω_i where $i = 1, 2, 3$. Following Kato [2] we can prove easly that all sets $\Omega_1, \Omega_2, \Omega_3$ are open and dense almost everywhere on M.

Case 1. Let $Rank\alpha(p) = 1$. Then any Jacobi operator R_X has three equal eigenvalues at any point $p \in \Omega_1$, i.e.:

$$a(p) = b(p) = c(p). \tag{6}$$

and (M, g) is a space of a pointwise constant sectional curvature at p. According to the Shour's theorem we have that (M, g) is a space of constant sectional curvature and hence (M, g) is a globally Osserman manifold on Ω_1 [11] .

Case 2. If $Rank\alpha(p) = 2$, then any Jacobi operator R_X has two eigenvalues at any point $q \in \Omega_1$. Now we have $\alpha_{13}(q)\alpha_{32}(q)\alpha_{33}(q) = 0$ and as we said above two eigenvalues of Jacobi operator are equal, suppose

$$a(q) \neq b(q) = c(q) \tag{7}$$

Then system (5) defined in an open neighborhoud $V_q \subset \Omega_2$ has the form

$$\begin{aligned} X(a) &= (\mu + \nu)(a - b), \\ X(b) &= \mu(b - a) = \nu(b - a). \end{aligned} \tag{8}$$

From our assumption $X(J_3) = 0$ and from the last system we have

$$X(J_3) = b(a - b)(\mu + \nu) = 0.$$

From here and (7) we have $b(q) = 0$ or $(\mu + \nu)(q) = 0$.

In the first subcase when $b = 0$, from (8) it follows that $c = 0$ and now from any results in [4] we have that (M, g) is flat on V_q , which is a trivial case for a globally Osserman manifold.

In the subcase or $(\mu + \nu)(q) = 0$ from the system (8) we have $X(a) = 0$ and then $X(b + c) = 0$ on V_q. Now from the second row of (8) we get $X(b) = -\mu a = -\nu a$, hence $X(b) = (\mu - \nu)a = 0$ at q . Further from (8) we have $(\mu - \nu)(q) = 0$ at q and hence $\mu(q) = \nu(q) = 0$ at q. Then from the system (5) it follows that $X(a) = X(b) = X(c) = 0$ at any point $q \in M$ which means that a, b, c are a globally constants and hence (M, g) is a globally Osserman manifold.

Case 3. If $Rank\alpha(p) = 3$, then any Jacobi operator R_X has three eigenvalues different from zero on a neighborhood (say U_p) at a point $p \in \Omega_3$. In this case we have

$\alpha_{13}(p)\alpha_{32}(p)\alpha_{33}(p) \neq 0$ and it is possible to write the well-known relation (6):

$$\frac{\varphi(p;X)}{\alpha_{31}(p)} = \frac{\psi(p;X)}{\alpha_{32}(p)} = \frac{\theta(p;X)}{\alpha_{33}(p)} \tag{9}$$

Now from (9) using the quaternionic transformation (3) we obtain:

$$\begin{array}{lll}
\varphi(p;X) = g(\nabla_A B, C), & \psi(p;X) = g(\nabla_A C, A), & \theta(p;X) = g(\nabla_C A, B), \\
\varphi(p;A) = g(\nabla_X C, B), & \psi(p;A) = g(\nabla_C B, X), & \theta(p;A) = g(\nabla_B X, C), \\
\varphi(p;B) = g(\nabla_C X, A), & \psi(p;B) = g(\nabla_X A, C), & \theta(p;B) = g(\nabla_X C, B), \\
\varphi(p;C) = g(\nabla_B A, X), & \psi(p;C) = g(\nabla_A X, B), & \theta(p;C) = g(\nabla_X B, A),
\end{array}$$

$$\begin{array}{rl}
\varphi(p;aX+bA) = & -a^3 g(\nabla_A B, C) + a^2 b g(\nabla_X B, C) + \\
& +a^2 b(g(\nabla_A B, C) - g(\nabla_X B, C)) - b^3 g(\nabla_X B, C), \\
\psi(p;aX+bA) = & a^3 g(\nabla_B C, A) + b^3 g(\nabla_C B, X) + \\
& +a^2 b(g(\nabla_B B, A) - g(\nabla_B C, X) - g(\nabla_C C, A)) + \\
& +ab^2(g(\nabla_C C, X) - g(\nabla_C B, A) - g(\nabla_B B, X)), \\
\theta(p;aX+bA) = & a^3 g(\nabla_C A, B) + \\
& a^2 b(g(\nabla_B A, B) - g(\nabla_C X, B) + g(\nabla_C C, A)) + \\
& +ab^2(g(\nabla_B A, B) - g(\nabla_B A, C) - g(\nabla_C C, X)) + \\
& +b^3 g(\nabla_B X, C).
\end{array} \tag{10}$$

Using (9) after a substitutions of X by A, B, C and having in mind (10) we get

$$\begin{array}{c}
\frac{g(\nabla_A B, C)}{\alpha_{31}(p)} = \frac{g(\nabla_B C, A)}{\alpha_{32}(p)} = \frac{g(\nabla_C A, B)}{\alpha_{33}(p)}, \\
\frac{g(\nabla_X C, B)}{\alpha_{31}(p)} = \frac{g(\nabla_C B, X)}{\alpha_{32}(p)} = \frac{g(\nabla_B X, C)}{\alpha_{33}(p)}, \\
\frac{g(\nabla_C X, A)}{\alpha_{31}(p)} = \frac{g(\nabla_X A, C)}{\alpha_{32}(p)} = \frac{g(\nabla_A C, X)}{\alpha_{33}(p)}, \\
\frac{g(\nabla_B A, X)}{\alpha_{31}(p)} = \frac{g(\nabla_A X, B)}{\alpha_{32}(p)} = \frac{g(\nabla_X B, A)}{\alpha_{33}(p)},
\end{array} \tag{11}$$

where $\alpha_{ij}(p)(i,j = 1,2,3,4)$ are a minors of $\alpha(p)$. Further we apply (5) for a tangent vector $aX + bA$ where a and b are an arbitrary real numbers such that $a^2 + b^2 = 1$. According to (10) and using (11) we obtain

$$\begin{array}{l}
a^2 b(\alpha_{32}(p) g(\nabla_X C, B) - \alpha_{31}(p)(g(\nabla_B B, A) - g(\nabla_C C, A) - g(\nabla_B C, X))) + \\
+ab^2(\alpha_{32}(p) g(\nabla_A B, C) - \alpha_{31}(p)(g(\nabla_C C, X) - g(\nabla_B B, X) - g(\nabla_C A, B))) = 0.
\end{array}$$

From this equality and (9) according to the denotions above we have:

$$\alpha_{31}(p)(\varphi - \theta - \nu + \mu) = 0$$

Applying (9) for the tangent vector $aX + bB$ and $aX + bC$ we obtain

$$\begin{aligned}&\alpha_{32}(p)(\psi - \varphi + \nu - \lambda) = 0,\\&\alpha_{33}(p)(\theta - \psi + \lambda - \mu) = 0\end{aligned}$$

and hence we have the system:

$$\begin{aligned}&\alpha_{31}(p)(\varphi - \theta - \nu + \mu) = 0,\\&\alpha_{32}(p)(\psi - \varphi + \nu - \lambda) = 0,\\&\alpha_{33}(p)(\theta - \psi + \lambda - \mu) = 0\end{aligned}$$

Because of a minors $\alpha_{31}(p), \alpha_{32}(p), \alpha_{33}(p)$ are different from zero then we have the system:

$$\begin{aligned}&\varphi - \theta - \nu + \mu = 0,\\&\psi - \varphi + \nu - \lambda = 0,\\&\theta - \psi + \lambda - \mu = 0.\end{aligned} \tag{12}$$

First we consider the equality $\varphi - \theta - \nu + \mu = 0$ or

$$g(\nabla_A B, C) - g(\nabla_C A, B) = g(\nabla_C C, X) - g(\nabla_B B, X).$$

Changing in this equality X by $aX + bA$ and using (3) we obtain:

$$\begin{aligned}&a^3(g(\nabla_A B, C) - g(\nabla_C X, B) - g(\nabla_C C, X) - g(\nabla_B B, X))+\\&+b^3(g(\nabla_X C, B) - g(\nabla_B A, C) - g(\nabla_B B, A) - g(\nabla_C C, A))+\\&+a^2b(-g(\nabla_X B, C) - g(\nabla_B X, B) + g(\nabla_C X, C) - g(\nabla_C A, B)-\\&-g(\nabla_C C, X) - g(\nabla_C B, X) - g(\nabla_C C, A)+\\&+g(\nabla_B B, A) - g(\nabla_B C, X) - g(\nabla_B B, X))+\\&+ab^2(-g(\nabla_A C, B) + g(\nabla_B X, C) - g(\nabla_B A, B) - g(\nabla_C A, C)-\\&-g(\nabla_B B, X) - g(\nabla_B C, A) - g(\nabla_C B, A) - g(\nabla_B C, B)+\\&g(\nabla_C B, A) - g(\nabla_C C, X)) = 0.\end{aligned} \tag{13}$$

From here we get

$$\begin{aligned}&g(\nabla_A B, C) - g(\nabla_C A, B) - g(\nabla_C C, X) + g(\nabla_B B, X) = 0,\\&g(\nabla_X C, B) - g(\nabla_B A, C) - g(\nabla_B B, A) - g(\nabla_C C, A) = 0.\end{aligned} \tag{14}$$

Using (3) we can either check that these equalities are equivalents. From (13) we have also

$$\begin{aligned}&g(\nabla_X B, C) + g(\nabla_B B, X) - g(\nabla_C C, X) + g(\nabla_C A, B)-\\&-2g(\nabla_B C, X) - 2g(\nabla_C B, X) - g(\nabla_C C, A) + g(\nabla_B B, A) = 0,\\[1ex]&g(\nabla_X C, B) - g(\nabla_B X, C) + g(\nabla_B B, A) - g(\nabla_C C, A)-\\&-2g(\nabla_B C, A) - 2g(\nabla_C B, A) - g(\nabla_C C, X) - g(\nabla_B B, X) = 0\end{aligned} \tag{15}$$

and using (3) we can see also that the last two equalities are equivalents. Thus from (14) and (15) we obtain the system

$$
\begin{aligned}
&g(\nabla_A B, C) - g(\nabla_C X, B) - g(\nabla_C C, X) - g(\nabla_B B, X) = 0,\\
&-g(\nabla_X B, C) + g(\nabla_B B, X) - g(\nabla_C C, X) - g(\nabla_C A, B)-\\
&-2g(\nabla_B C, X) - 2g(\nabla_C B, X) - g(\nabla_C C, A) - g(\nabla_B B, A) = 0,
\end{aligned}
\tag{16}
$$

and from here we have

$$
g(\nabla_C A, B) - g(\nabla_C X, B) = 0. \tag{17}
$$

Replasing in this system X by $aX + bA$ and using (3) we obtain the equation:

$$
\begin{aligned}
&a^3(g(\nabla_C X, B) - g(\nabla_C A, B)) + b^3(g(\nabla_B X, C) + g(\nabla_B A, C))+\\
&+a^2b(-g(\nabla_B B, X) + g(\nabla_B B, A) + g(\nabla_C C, A)+\\
&+g(\nabla_C A, B) + g(\nabla_C X, B) - g(\nabla_C C, A))+\\
&+ab^2(-g(\nabla_B B, X) - g(\nabla_B X, C) - g(\nabla_B B, A)+\\
&+g(\nabla_B A, C) + g(\nabla_C X, X) - g(\nabla_C C, A)) = 0.
\end{aligned}
$$

We sum the coefficients before a^2b and ab^2 which are vanishing and so we obtain the equality:

$$
\begin{aligned}
&-2g(\nabla_B B, X) + g(\nabla_C C, X) + g(\nabla_C A, B)+\\
&+g(\nabla_C X, B) - g(\nabla_B X, C) + g(\nabla_B A, C) = 0.
\end{aligned}
$$

Since the coefficients before a^3 and b^3 are vanishing, then

$$
\begin{aligned}
g(\nabla_C X, B) + g(\nabla_C A, B) = 0,\\
g(\nabla_B X, C) + g(\nabla_B A, C) = 0
\end{aligned}
$$

and from here we have $-\mu + \nu + \theta - \varphi = 0$. Because from the results above we have $-\mu + \nu + \theta - \psi = 0$, then summing the last two equalities we obtain $\varphi = \psi$. Analogously changing in (17) X by $aX + bB$ and having in mind (3) we obtain $\varphi = \theta$. Finally we have $\varphi = \psi = \theta$ and then the system (4) has the form:

$$
\begin{aligned}
\varphi(2a - b - c) = 0,\\
\psi(2b - c - a) = 0,\\
\theta(2c - a - b) = 0.
\end{aligned}
\tag{18}
$$

If $\varphi(p; X) \neq 0$, then we obtain (6) which is not possible when $p \in \Omega_3$, hence $\varphi(p; X) = 0$. Then $\varphi = \psi = \theta$ and now the system (5) has the form:

(19)
$$\begin{aligned} X(a) &= \lambda(2a-b-c),\\ X(b) &= \lambda(2b-c-a),\\ X(c) &= \lambda(2c-a-b), \end{aligned}$$

for any tangent vector $X \in \Omega_3$. This expression of the system (5) which follows from the assumption $Rank\alpha(p) = 3$, contradict with our hypothesis (M, g) to be a pointwise Osserman on Ω_3 and now we prove this fact. Since J_1 is a globally constant on M then $X(J_1) = 0$ and from the system (5) it follows that:

(20)
$$X(J_2) = \lambda((a-b)^2 + (a-c)^2 + (b-c)^2).$$

Nowfrom the Viet-formulas:

(21)
$$\begin{aligned} J_1 - \sigma_1 &= 0,\\ J_2 - J_1\sigma_1 + 2\sigma_1 &= 0,\\ J_3 - J_2\sigma_1 + J_1\sigma_2 - 3\sigma_1 &= 0, \end{aligned}$$

we obtain $X(-J_1\sigma_1 + 2\sigma_1) = X(J_2) = 0$ which is not possible when $p \in \Omega_3$ and it was proved in [8].

Finally we remark that if J_1 is a pointwise constant at any point $p \in M$ and if $J_3 = 0$, then (M, g) is a reducible spase or (M, g) is flat [8]. Hence this result complete Theorem 2.

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