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STUDING SPECIAL TRIANGLES BY SHAPES

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A shape of triangle is a complex number which is an equivalence class of triangles with respect to the equivalence relation a direct similarity. In this paper we apply shapes for examining equilateral and right triangles in several examples.

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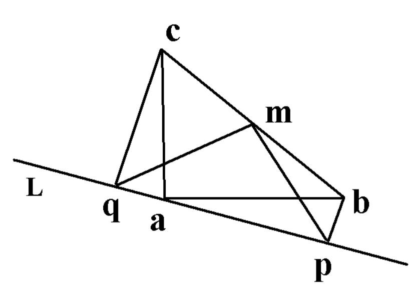
It is well-knoun that the complex numbers can be used for studing of Euclidean plane. There are books which consider applications of complex cross-ratio in plane geometry (for example see [3] and [8]). A complex analytic formalism based on complex cross-ratio is developed by June Lester in her triangle series ([4], [5] and [6]). The concept of shape of triangle is a very useful tool in this formalism. Some applications of shapes of triangles are given in [1] and [2]. In the paper, we apply shapes for establishment of the type of triangle. Note that another interpretation of the shape of triangle is introduced by H. Sato in [7].

In according to [4], if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three noncollinear points in the Gaussian plane, the number

$$\triangle_{\mathbf{abc}} = \frac{\mathbf{a}-\mathbf{c}}{\mathbf{a}-\mathbf{b}} \quad \in \quad \mathbb{C} \setminus \mathbb{R}$$

is called a shape of triangle $\triangle\mathbf{abc}$. The ratio of the side lengths $|\mathbf{a}-\mathbf{c}|$ and $|\mathbf{a}-\mathbf{b}|$ is equal to $|\triangle_{\mathbf{abc}}|$. A positive oriented equilateral triangle $\triangle\mathbf{abc}$ has a shape $\triangle_{\mathbf{abc}} = \omega = \frac{1}{2}+i\frac{\sqrt{3}}{2}$. Similarly, a negative oriented equilateral triangle $\triangle\mathbf{abc}$ has a shape $\triangle_{\mathbf{abc}} = \overline{\omega} = \frac{1}{2}-i\frac{\sqrt{3}}{2}$. A triangle $\triangle\mathbf{abc}$ is rightangled at $\mathbf{a}$ if and only if its shape $\triangle_{\mathbf{abc}}$ is pure imaginary. We shall also apply the Second Shape Theorem and the angle-shape formula (see for details [4]).

Proposition 1 *Let* $\triangle\mathbf{abc}$ *be an isosceles and right triangle with a right angle at* $\mathbf{a}$. *Let* $\mathbf{L}$ *be a line passing through* $\mathbf{a}$ *and not intersecting the segment* $\mathbf{bc}$. *If* $\mathbf{p}$, $\mathbf{q}$ *are distinct points on the line* $\mathbf{L}$ *such that* $\mathbf{bp}$ *and* $\mathbf{cq}$ *are perpendicular to* $\mathbf{L}$ *and if* $\mathbf{m}$ *is the midpoint of the segment* $\mathbf{bc}$, *then the triangle* $\triangle\mathbf{mqp}$ *is isosceles and right.*

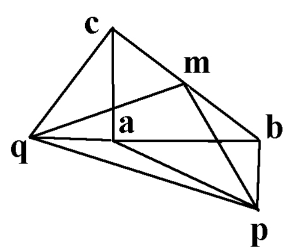


Proof. The shape of $\triangle\mathbf{abc}$ is $\triangle_{\mathbf{abc}} = i$. From the similarity of the triangles $\triangle\mathbf{qac}$ and $\triangle\mathbf{pba}$, it follows that $\triangle_{\mathbf{qac}} = \triangle_{\mathbf{pba}} = \lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $\mathbf{m}$ is the midpoint of the segment $\mathbf{bc}$, $[\infty, \mathbf{m}; \mathbf{c}, \mathbf{b}] = -1$. Using the Second Shape Theorem from [4] we have

$$\triangle_{\mathbf{mqp}} = \frac{-i - \frac{1+\lambda}{1-\lambda}}{\frac{1+\lambda}{1-\lambda}i - 1} = \frac{i(\frac{1+\lambda}{1-\lambda}i - 1)}{\frac{1+\lambda}{1-\lambda}i - 1} = i.$$

This means that $\angle\mathbf{pmq} = \pi : 2$ and $\triangle\mathbf{mqp}$ is an isosceles with apex at $\mathbf{m}$.

Proposition 2 *On the sides* $\mathbf{ac}$ *and* $\mathbf{ab}$ *of* $\triangle\mathbf{abc}$*, construct similar triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{pba}$ *with the same orientation. Let* $\mathbf{m}$ *be the midpoint of the segment* $\mathbf{bc}$ *and let* $\mathbf{m}$*,* $\mathbf{q}$ *and* $\mathbf{p}$ *be distinct. Then* $\triangle\mathbf{mqp}$ *is similar to* $\triangle\mathbf{abc}$ *if and only if* $\triangle\mathbf{abc}$ *is isosceles and right.*



Proof. Let the shape of the triangle $\triangle\mathbf{abc}$ is $\triangle$, i.e. $\triangle = \triangle_{\mathbf{abc}}, \quad \triangle \in \mathbb{C} \setminus \mathbb{R}$. From the similarity of the triangles $\triangle_{\mathbf{qac}}$ and $\triangle_{\mathbf{pba}}$, it follows that $\triangle_{\mathbf{qac}} = \triangle_{\mathbf{pba}} = \lambda, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}$. Using Second Shape Theorem from [4] and $[\infty, \mathbf{m}; \mathbf{c}, \mathbf{b}] = -1$, we obtain the shape of the triangle $\triangle\mathbf{mqp}$, i.e.

$$\triangle_{\mathbf{mqp}} = \frac{-\triangle - \frac{1+\lambda}{1-\lambda}}{\frac{1+\lambda}{1-\lambda}\triangle - 1}.$$

A necessary and sufficient condition for a similarity of the triangles $\triangle\mathbf{mqp}$ and $\triangle\mathbf{abc}$ is $\triangle_{\mathbf{mqp}} = \triangle_{\mathbf{abc}} = \triangle$, i.e.

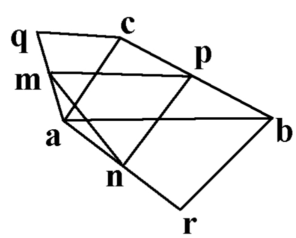
$$\frac{-\triangle - \frac{1+\lambda}{1-\lambda}}{\frac{1+\lambda}{1-\lambda}\triangle - 1} = \triangle.$$

The last equality is equivalent to the equality $\triangle^2 + 1 = 0$. Whence either $\triangle = i$ or $\triangle = -i$ and the proof is completed.

The advantage of the Second Shape Theorem is that it gives an equality for shapes of five triangles. In the next three propositions, we establish equalities for shapes in some other configurations of triangles. These equations can be consider as analogues of the Second Shape Theorem. Moreover, we apply any obtained equality for configurations with special triangles.

Proposition 3 *On the sides* **ac** *and* **ba** *of* $\triangle\mathbf{abc}$*, construct triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with shapes* μ *and* ν*, respectively. If the points* **m**, **n** *and* **p** *are the midpoints of the segments* **aq**, **ar** *and* **cb**, *respectively, then* $\triangle\mathbf{pmn}$ *has a shape*

$$\triangle_{\mathbf{pmn}} = \frac{\frac{1}{1-\nu} + \triangle_{\mathbf{abc}}}{1 + \frac{\mu}{\mu-1}\triangle_{\mathbf{abc}}}. \tag{1}$$



Proof. Solve the equations $\triangle_{\mathbf{qac}} = \mu$ and $\triangle_{\mathbf{rba}} = \nu$ for **q** and **r**: $\mathbf{q} = \frac{\mathbf{c}-\mu\mathbf{a}}{1-\mu}$, $\mathbf{r} = \frac{\mathbf{a}-\nu\mathbf{b}}{1-\nu}$.
Then calculate
$2(1-\nu)(\mathbf{p}-\mathbf{n}) = (\nu-2)\mathbf{a} + \mathbf{b} + (1-\nu)\mathbf{c} = -(\mathbf{a}-\mathbf{b}) - (1-\nu)(\mathbf{a}-\mathbf{c}) =$
$= -(\mathbf{a}-\mathbf{b})[1 + (1-\nu)\triangle_{\mathbf{abc}}]$,
$2(1-\mu)(\mathbf{p}-\mathbf{m}) = (2\mu-1)\mathbf{a} + (1-\mu)\mathbf{b} - \mu\mathbf{c} = (\mu-1)(\mathbf{a}-\mathbf{b}) + \mu(\mathbf{a}-\mathbf{c}) =$
$= (\mathbf{a}-\mathbf{b})[\mu - 1 + \mu\triangle_{\mathbf{abc}}]$.
By division we get

$$\triangle_{\mathbf{pmn}} = \frac{[1 + (1-\nu)\triangle_{\mathbf{abc}}](\mu-1)}{(\mu - 1 + \mu\triangle_{\mathbf{abc}})(1-\nu)}.$$

Corollary 3.1 *On the sides* **ac** *and* **ba** *of* $\triangle\mathbf{abc}$*, construct equilateral triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with the same orientation. If the points* **m**, **n** *and* **p** *are midpoints of the segments* **aq**, **ar** *and* **cb**, *respectively, then the triangle* $\triangle\mathbf{pmn}$ *is also equilateral.*

Proof. From Proposition 3, we have that $\mu = \nu = \omega$ or $\mu = \nu = \overline{\omega}$. If $\mu = \nu = \omega$ we obtain that $\triangle_{\mathbf{pmn}} = \frac{1+(1-\omega)\triangle_{\mathbf{abc}}}{1-\omega-\omega\triangle_{\mathbf{abc}}}$. Then using the relations $\omega\overline{\omega} = 1$, $\omega + \overline{\omega} = 1$ and $\overline{\omega}^2 - \overline{\omega} + 1 = 0$ we calculate

$$\triangle_{\mathbf{pmn}} = \frac{1 + \overline{\omega}\triangle_{\mathbf{abc}}}{\overline{\omega} - \omega\triangle_{\mathbf{abc}}} = \frac{\omega(\overline{\omega} + \overline{\omega}^2\triangle_{\mathbf{abc}})}{\overline{\omega} - \omega\triangle_{\mathbf{abc}}} = \frac{\omega[\overline{\omega} + (\overline{\omega}-1)\triangle_{\mathbf{abc}}]}{\overline{\omega} - \omega\triangle_{\mathbf{abc}}} = \frac{\omega(\overline{\omega} - \omega\triangle_{\mathbf{abc}})}{\overline{\omega} - \omega\triangle_{\mathbf{abc}}} = \omega,$$

i.e. $\triangle\mathbf{pmn}$ is equilateral. The same conclusion hold if $\mu = \nu = \overline{\omega}$ (replace ω by $\overline{\omega}$).

Corollary 3.2 *On the sides* **ac** *and* **ba** *of* $\triangle\mathbf{abc}$ *construct similar triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with corresponding vertices distinct. Let* **m**, **n** *and* **p** *be the midpoints of the segments* **aq**, **ar** *and* **bc**, *respectively. Then* $\triangle\mathbf{pmn}$ *is similar to* $\triangle\mathbf{qac}$ *if and only if both triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *are equilateral.*

Proof. Let the shape of the triangle $\triangle\mathbf{qac}$ is $\lambda \in \mathbb{C} \setminus \mathbb{R}$. From the similarity of the triangles $\triangle\mathbf{qac}$ and $\triangle\mathbf{rba}$, it follows that $\triangle_{\mathbf{qac}} = \triangle_{\mathbf{rba}} = \lambda$. From Proposition 3 we have $\mu = \nu = \lambda$ and

$$\triangle_{\mathbf{pmn}} = \frac{1 + (1-\lambda)\triangle_{\mathbf{abc}}}{1 - \lambda - \lambda\triangle_{\mathbf{abc}}}.$$

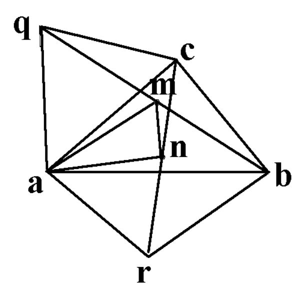
A necessary and sufficient condition for a similarity of the triangles $\triangle\mathbf{pmn}$ and $\triangle\mathbf{qab}$ is $\triangle_{\mathbf{pmn}} = \triangle_{\mathbf{qac}} = \lambda$, i.e.

$$\frac{1 + (1-\lambda)\triangle_{\mathbf{abc}}}{1 - \lambda - \lambda\triangle_{\mathbf{abc}}} = \lambda.$$

From here, we obtain that $\lambda^2 - \lambda + 1 = 0$, i.e. either $\lambda = \omega$ or $\lambda = \overline{\omega}$.

Proposition 4 *On the sides* $\mathbf{ac}$ *and* $\mathbf{ba}$ *of* $\triangle\mathbf{abc}$ *construct triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with shapes* μ *and* ν*, respectively. If the points* $\mathbf{m}$ *and* $\mathbf{n}$ *are midpoints of the segments* $\mathbf{bq}$ *and* $\mathbf{cr}$ *then* $\triangle\mathbf{anm}$ *has a shape*

$$\triangle_{\mathbf{anm}} = \frac{1 + \frac{1}{1-\mu}\triangle_{\mathbf{abc}}}{-\frac{\nu}{1-\nu} + \triangle_{\mathbf{abc}}}. \tag{2}$$



Proof. We have that $\triangle_{\mathbf{qac}} = \mu$ and $\triangle_{\mathbf{rba}} = \nu$, i.e. $\mathbf{q} = \frac{\mathbf{c}-\mu\mathbf{a}}{1-\mu}$ and $\mathbf{r} = \frac{\mathbf{a}-\nu\mathbf{b}}{1-\nu}$. Then calculate

$2(1-\mu)(\mathbf{a}-\mathbf{m}) = (2-\mu)\mathbf{a} - (1-\mu)\mathbf{b} - \mathbf{c} = (1-\mu)(\mathbf{a}-\mathbf{b}) + \mathbf{a} - \mathbf{c} =$
$= (\mathbf{a}-\mathbf{b})(1-\mu+\triangle_{\mathbf{abc}}),$
$2(1-\nu)(\mathbf{a}-\mathbf{n}) = (1-2\nu)\mathbf{a} + \nu\mathbf{b} + (1-\nu)\mathbf{c} = -\nu(\mathbf{a}-\mathbf{b}) + (1-\nu)(\mathbf{a}-\mathbf{c}) =$
$= (\mathbf{a}-\mathbf{b})[-\nu + (1-\nu)\triangle_{\mathbf{abc}}].$

Divide and rearrange to get

$$\triangle_{\mathbf{anm}} = \frac{(1-\mu+\triangle_{\mathbf{abc}})(1-\nu)}{(-\nu+(1-\nu)\triangle_{\mathbf{abc}})(1-\mu)}.$$

Corollary 4.1 *On the sides* $\mathbf{ac}$ *and* $\mathbf{ba}$ *of* $\triangle\mathbf{abc}$*, construct equilateral triangles* $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with the same orientation. If the points* $\mathbf{m}$ *and* $\mathbf{n}$ *are midpoints on the segments* $\mathbf{bq}$ *and* $\mathbf{cr}$*, respectively, then* $\triangle\mathbf{anm}$ *is equilateral.*

Proof. We may assume that $\mu = \nu = \omega$ in Proposition 4. Then

$$\triangle_{\mathbf{anm}} = \frac{1-\omega+\triangle_{\mathbf{abc}}}{-\omega+(1-\omega)\triangle_{\mathbf{abc}}} = \frac{\overline{\omega}+\triangle_{\mathbf{abc}}}{-\omega+\overline{\omega}\triangle_{\mathbf{abc}}}.$$

From the relations $\overline{\omega}^2 - \overline{\omega} + 1 = 0$ and $\omega + \overline{\omega} = 1$, it follows that $\overline{\omega}^2 = -\omega$. Hence,

$$\triangle_{\mathbf{anm}} = \frac{\omega(\overline{\omega}^2+\overline{\omega}\triangle_{\mathbf{abc}})}{-\omega+\overline{\omega}\triangle_{\mathbf{abc}}} = \omega,$$

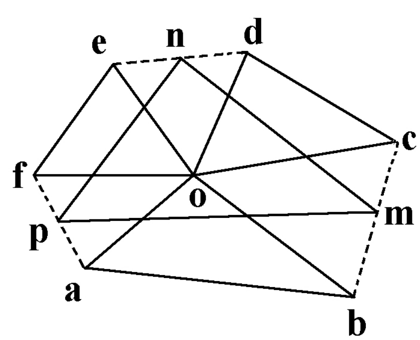
i.e. $\triangle\mathbf{anm}$ is equilateral.

Corollary 4.2 *On the sides* **ac** *and* **ab** *of the triangle* $\triangle\mathbf{abc}$ *construct similar triangles* $\triangle\mathbf{acq}$ *and* $\triangle\mathbf{arb}$*. Let* **m** *and* **n** *be the midpoint of the segments* **bq** *and* **cr***, respectively. Then the triangle* $\triangle\mathbf{anm}$ *is similar to the triangles* $\triangle\mathbf{acq}$ *and* $\triangle\mathbf{arb}$*.*

Proof. From the similarity of the triangles $\triangle_{\mathbf{acq}}$ and $\triangle_{\mathbf{arb}}$, it follows that $\triangle_{\mathbf{acq}} = \triangle_{\mathbf{arb}} = \lambda$. Using Proposition 4 we have that $\mu = \triangle_{\mathbf{qac}} = \triangle''_{\mathbf{acq}} = \lambda''$ and $\nu = \triangle_{\mathbf{rba}} = \triangle'_{\mathbf{arb}} = \lambda'$. Replacing, we obtain $\triangle_{\mathbf{anm}} = \frac{1+\frac{1}{1-\lambda''}\triangle_{\mathbf{abc}}}{-\frac{\lambda'}{1-\lambda'}+\triangle_{\mathbf{abc}}}$. From $\lambda' = \frac{1}{1-\lambda}$, $\lambda'' = 1-\frac{1}{\lambda}$ we find that $\triangle_{\mathbf{anm}} = \frac{1+\lambda\triangle_{\mathbf{abc}}}{\frac{1}{\lambda}+\triangle_{\mathbf{abc}}} = \lambda$. Consequently, the triangles $\triangle\mathbf{anm}$, $\triangle\mathbf{acq}$ and $\triangle\mathbf{arb}$ are similar.

Proposition 5 *Let* $\triangle\mathbf{oab}$*,* $\triangle\mathbf{ocd}$ *and* $\triangle\mathbf{oef}$ *be triangles with shapes* λ*,* μ *and* ν*, respectively. If* **m***,* **n** *and* **p** *are midpoints of the segments* **bc***,* **de** *and* **fa***, respectively, then the triangle* $\triangle_{\mathbf{pmn}}$ *has a shape*

$$\triangle\mathbf{pmn} = \frac{\triangle_{\mathbf{oea}} - \mu\triangle_{\mathbf{oec}} + \nu - 1}{(1-\lambda)\triangle_{\mathbf{oea}} - \triangle_{\mathbf{oec}} + \nu}. \tag{3}$$



Proof. From $\lambda = \triangle_{\mathbf{oab}}$, $\mu = \triangle_{\mathbf{ocd}}$, $\nu = \triangle_{\mathbf{oef}}$ we have that

$$\mathbf{o}-\mathbf{b} = \lambda(\mathbf{o}-\mathbf{a}), \mathbf{o}-\mathbf{d} = \mu(\mathbf{o}-\mathbf{c}), \mathbf{o}-\mathbf{f} = \nu(\mathbf{o}-\mathbf{e}).$$

Sinse $\mathbf{p} = \frac{\mathbf{a}+\mathbf{f}}{2}$, $\mathbf{m} = \frac{\mathbf{b}+\mathbf{c}}{2}$, $\mathbf{n} = \frac{\mathbf{d}+\mathbf{e}}{2}$ we obtain

$$\begin{aligned}\triangle_{\mathbf{pmn}} &= \frac{\mathbf{p}-\mathbf{n}}{\mathbf{p}-\mathbf{m}} = \frac{\mathbf{a}+\mathbf{f}-\mathbf{d}-\mathbf{e}}{\mathbf{a}+\mathbf{f}-\mathbf{b}-\mathbf{c}} = \frac{(\mathbf{a}-\mathbf{o})+(\mathbf{f}-\mathbf{o})-(\mathbf{d}-\mathbf{o})-(\mathbf{e}-\mathbf{o})}{(\mathbf{a}-\mathbf{o})+(\mathbf{f}-\mathbf{o})-(\mathbf{b}-\mathbf{o})-(\mathbf{c}-\mathbf{o})} = \\ &= \frac{(\mathbf{a}-\mathbf{o})+(\nu-1)(\mathbf{e}-\mathbf{o})-\mu(\mathbf{c}-\mathbf{o})}{(1-\lambda)(\mathbf{a}-\mathbf{o})+\nu(\mathbf{e}-\mathbf{o})-(\mathbf{c}-\mathbf{o})} = \frac{\triangle_{\mathbf{oea}}-\mu\triangle_{\mathbf{oec}}+\nu-1}{(1-\lambda)\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}+\nu}.\end{aligned}$$

Corollary 5.1 *Let* $\triangle\mathbf{oab}$*,* $\triangle\mathbf{ocd}$ *and* $\triangle\mathbf{oef}$ *be equilateral triangles with the same orientation. If* $\mathbf{m}$*,* $\mathbf{n}$ *and* $\mathbf{p}$ *are midpoints of the segments* $\mathbf{bc}$*,* $\mathbf{de}$ *and* $\mathbf{fa}$*, respectively, then the triangle* $\triangle\mathbf{mnp}$ *is equilateral.*

Proof. Replacing $\lambda = \mu = \nu = \omega$ in Proposition 5, we obtain that

$$\triangle_{\mathbf{pmn}} = \frac{\triangle_{\mathbf{oea}}-\omega\triangle_{\mathbf{oec}}+\omega-1}{(1-\omega)\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}+\omega}.$$

Using the relations $\omega^2-\omega+1=0$, $\omega\overline{\omega}=1$ and $\omega+\overline{\omega}=1$, we have that

$$\triangle_{\mathbf{pmn}} = \frac{\omega(\overline{\omega}\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}+1-\overline{\omega})}{\overline{\omega}\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}+1-\overline{\omega}} = \omega.$$

Corollary 5.2 *Let* $\triangle\mathbf{oab}$*,* $\triangle\mathbf{ocd}$ *and* $\triangle\mathbf{oef}$ *be similar triangles. Let* $\mathbf{m}$*,* $\mathbf{n}$ *and* $\mathbf{p}$ *be the midpoints of the segments* $\mathbf{bc}$*,* $\mathbf{de}$ *and* $\mathbf{fa}$ *, respectively. Then the triangle* $\triangle\mathbf{pmn}$ *is similar to the triangle* $\triangle\mathbf{oab}$ *if and only if* $\triangle\mathbf{oab}$*,* $\triangle\mathbf{ocd}$ *and* $\triangle\mathbf{oef}$ *are equilateral.*

Proof. From similarity of the triangles $\triangle\mathbf{oab}$, $\triangle\mathbf{ocd}$ and $\triangle\mathbf{oef}$, it follows that $\triangle_{\mathbf{oab}} = \triangle_{\mathbf{ocd}} = \triangle_{\mathbf{oef}}$. Hence, setting $\lambda=\mu=\nu$ into the formula of Proposition 5 we have that

$$\begin{aligned}\triangle_{\mathbf{pmn}} &= \frac{\triangle_{\mathbf{oea}}-\lambda\triangle_{\mathbf{oec}}+\lambda-1}{(1-\lambda)\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}+\lambda} = \frac{\triangle_{\mathbf{oea}}-1-\lambda(\triangle_{\mathbf{oec}}-1)}{\triangle_{\mathbf{oea}}-\triangle_{\mathbf{oec}}-\lambda(\triangle_{\mathbf{oea}}-1)} = \\ &= \frac{\mathbf{e}-\mathbf{a}-\lambda(\mathbf{e}-\mathbf{c})}{\mathbf{c}-\mathbf{a}-\lambda(\mathbf{e}-\mathbf{a})} = \frac{\triangle_{\mathbf{ace}}-\lambda(\triangle_{\mathbf{ace}}-1)}{1-\lambda\triangle_{\mathbf{ace}}}.\end{aligned}$$

A necessary and suficient condition for a similarity of the triangles $\triangle\mathbf{pmn}$ and $\triangle\mathbf{oab}$ is $\triangle_{\mathbf{pmn}} = \triangle_{\mathbf{oab}} = \lambda$, i.e.

$$\frac{\triangle_{\mathbf{ace}}-\lambda(\triangle_{\mathbf{ace}}-1)}{1-\lambda\triangle_{\mathbf{ace}}} = \lambda,$$

wich from $\triangle_{\mathbf{ace}} \neq 0$ is equivalent to the equality $\lambda^2-\lambda+1=0$. From here either $\lambda=\omega$ or $\lambda=\overline{\omega}$ and the proof is completed.

Corollary 5.3 *Let* $\triangle\mathbf{oab}$*,* $\triangle\mathbf{ocd}$ *and* $\triangle\mathbf{oef}$ *be similar triangles. Let* $\mathbf{m}$*,* $\mathbf{n}$ *and* $\mathbf{p}$ *be the midpoints of the segments* $\mathbf{bc}$*,* $\mathbf{de}$ *and* $\mathbf{fa}$*, respectively. Then, the triangle* $\triangle\mathbf{pmn}$ *is similar to the triangle* $\triangle\mathbf{ace}$ *if and only if* $\triangle\mathbf{ace}$ *is equilateral.*

Proof. Using the proof of Corollary 5.2 we have that

$$\triangle_{\mathbf{pmn}} = \frac{\triangle_{\mathbf{ace}} - \lambda(\triangle_{\mathbf{ace}} - 1)}{1 - \lambda\triangle_{\mathbf{ace}}}.$$

A necessary and sufficient condition for a similarity of the triangles $\triangle\mathbf{pmn}$ and $\triangle\mathbf{ace}$ is $\triangle_{\mathbf{pmn}} = \triangle_{\mathbf{ace}}$, i.e.

$$\frac{\triangle_{\mathbf{ace}} - \lambda(\triangle_{\mathbf{ace}} - 1)}{1 - \lambda\triangle_{\mathbf{ace}}} = \triangle_{\mathbf{ace}}.$$

Since $\lambda \neq 0$ we obtain the equivalent equality $\triangle^2_{\mathbf{ace}} - \triangle_{\mathbf{ace}} + 1 = 0$ with solutions ω and $\overline{\omega}$.

Now, we apply the same technique in some other examples.

Example 1 *Let $\mathbf{aa}_1$ and $\mathbf{bb}_1$ be the altitudes of $\triangle\mathbf{abc}$ through the vertices $\mathbf{a}$ and $\mathbf{b}$. Let points $\mathbf{a}_2$ and $\mathbf{b}_2$ lie on the lines $\mathbf{aa}_1$ and $\mathbf{bb}_1$, respectively, such that $\mathbf{a}$ is between $\mathbf{a}_1$ and $\mathbf{a}_2$, $|\mathbf{a}_2 - \mathbf{a}| = |\mathbf{b} - \mathbf{c}|$, $\mathbf{b}$ is between $\mathbf{b}_1$ and $\mathbf{b}_2$, $|\mathbf{b}_2 - \mathbf{b}| = |\mathbf{c} - \mathbf{a}|$. Then the triangle $\triangle\mathbf{ca}_2\mathbf{b}_2$ is isosceles and right.*

Proof. From $\mathbf{a}_2\mathbf{a}\perp\mathbf{bc}$ and $|\mathbf{a}_2 - \mathbf{a}| = |\mathbf{b} - \mathbf{c}|$, it follows that $\frac{\mathbf{a}_2-\mathbf{a}}{\mathbf{b}-\mathbf{c}} = \pm i$. Similarly, we have that $\frac{\mathbf{b}_2-\mathbf{b}}{\mathbf{c}-\mathbf{a}} = \pm i$. Hence $\mathbf{a}_2 = \mathbf{a} \pm i(\mathbf{b} - \mathbf{c})$ and $\mathbf{b}_2 = \mathbf{b} \pm i(\mathbf{c} - \mathbf{a})$. Thus,

$$\triangle_{\mathbf{ca}_2\mathbf{b}_2} = \frac{\mathbf{c} - \mathbf{b}_2}{\mathbf{c} - \mathbf{a}_2} = \frac{(\mathbf{c} - \mathbf{b}) \mp i(\mathbf{c} - \mathbf{a})}{(\mathbf{c} - \mathbf{a}) \pm i(\mathbf{c} - \mathbf{b})} = i. \quad \text{Q.E.D.}$$

Example 2 *Let $\triangle\mathbf{abc}$ be triangle with $\angle\mathbf{bca} = 2\angle\mathbf{cab}$ and $|\mathbf{a} - \mathbf{c}| = 2|\mathbf{b} - \mathbf{c}|$. Then $\triangle\mathbf{abc}$ is right.*

Proof. Appling the angle-shape formula for the triangle $\triangle\mathbf{bca}$, we obtain its shape

$$(4) \qquad \triangle = \triangle_{\mathbf{bca}} = \frac{1 - e^{-2i(\angle\mathbf{bca})}}{1 - e^{2i(\angle\mathbf{cab})}} = \frac{1 - e^{-4iA}}{1 - e^{2iA}} = \frac{e^{-4iA}(e^{4iA} - 1)}{-(e^{2iA} - 1)} = -\frac{1 + e^{2iA}}{e^{4iA}}.$$

From $\triangle' = \triangle_{\mathbf{cab}} = \frac{|\mathbf{c}-\mathbf{b}|}{|\mathbf{c}-\mathbf{a}|}e^{i(\angle\mathbf{bca})} = \frac{1}{2}e^{2iA}$, it follows that $e^{2iA} = 2\triangle'$. Replacing in (4), we have $\triangle = -\frac{2\triangle'+1}{4\triangle'^2}$. But $\triangle' = \frac{1}{1-\triangle}$ and then $4\triangle = -2(1 - \triangle) - (1 - \triangle)^2$. Whence $\triangle^2 = -3$. Since the shape of the triangle $\triangle\mathbf{bca}$ is pure imaginary, it is right.More exactly, $\angle\mathbf{abc} = \pi : 2$, $\angle\mathbf{bca} = \pi : 3$ and $\angle\mathbf{cab} = \pi : 6$.

Example 3 *Let $\triangle\mathbf{abc}, \triangle\mathbf{cde}, \triangle\mathbf{ehk}$ be equilateral triangles with the same orientation such that $\overrightarrow{\mathbf{ad}} = \overrightarrow{\mathbf{dk}}$. Then the triangle $\triangle\mathbf{bhd}$ also is equilateral.*

Proof. If the shapes of triangles $\triangle\mathbf{abc}, \triangle\mathbf{cde}$ and $\triangle\mathbf{ehk}$ are ω, then $\frac{\mathbf{a}-\mathbf{c}}{\mathbf{a}-\mathbf{b}} = \frac{\mathbf{c}-\mathbf{e}}{\mathbf{c}-\mathbf{d}} = \frac{\mathbf{e}-\mathbf{k}}{\mathbf{e}-\mathbf{h}} = \omega$. From here $\mathbf{b} = \frac{\mathbf{c}+(\omega-1)\mathbf{a}}{\omega}$, $\mathbf{d} = \frac{\mathbf{e}+(\omega-1)\mathbf{c}}{\omega}$ and $\mathbf{h} = \frac{\mathbf{k}+(\omega-1)\mathbf{e}}{\omega}$. On the other hand **d-a** = **k-d**. Hence $\mathbf{k} = 2\mathbf{d} - \mathbf{a} = \frac{2(\omega-1)\mathbf{c}+2\mathbf{e}-\omega\mathbf{a}}{\omega}$. Thus, we obtain

$$\begin{aligned}\triangle_{\mathbf{bhd}} &= \frac{\mathbf{b} - \mathbf{d}}{\mathbf{b} - \mathbf{h}} = \frac{(\omega - 1)\mathbf{a} + (2 - \omega)\mathbf{c} - \mathbf{e}}{(\omega - 1)\mathbf{a} + \mathbf{c} - (\omega - 1)\mathbf{e} - \mathbf{k}} = \\ &= \frac{\omega[(\omega - 1)\mathbf{a} + (2 - \omega)\mathbf{c} - \mathbf{e}]}{\omega^2\mathbf{a} + (2 - \omega)\mathbf{c} - (\omega^2 - \omega + 2)\mathbf{e}} = \frac{\omega[\omega^2\mathbf{a} + (2 - \omega)\mathbf{c} - \mathbf{e}]}{\omega^2\mathbf{a} + (2 - \omega)\mathbf{c} - \mathbf{e}} = \omega\end{aligned}$$

(ω is the solution of the equation $\omega^2 - \omega + 1 = 0$), i.e. the triangle $\triangle\mathbf{bhd}$ is equilateral. We obtain the same conclusion if $\triangle_{\mathbf{abc}} = \triangle_{\mathbf{cde}} = \triangle_{\mathbf{ehk}} = \overline{\omega}$ (replace ω by $\overline{\omega}$).

The Second Shape Theorem as well our formulas (1), (2) and (3) hold for arbitrary triangles. This shows the meaning of the shape for studing of triangle geometry.

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