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INTEGRO-SUMMATION INEQUALITIES WITH A STEP FUNCTION AND APPLICATIONS

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Abstract. An integro-summation inequality of Gronwall-Bellman type is proved and applied for studying some qualitative properties of the solutions of differential equations with a step function.

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1. Introduction

One of the most powerful apparatuses in the quantative theory of differential equations is the integral inequalities of Gronwall-Bellman type [1], [2]. They are greatly used in investigating diverse properties of the solutions of the differential equations such as uniqueness, continuous dependence, stability, boundedness, etc.

In the paper we shall prove a linear integro-summation inequality of Gronwall-Bellman type and shall use this inequality for investigating some qualitative properties of the solutions of differential equations with a piecewise constant function (DEPCF). Some properties of the solutions of DEPCF are studied in [3], [4]. We note that research in this direction is motivated by the fact that DEPCF represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

2. Main results

Let the points $t_k \in \mathbb{R}, k = 0, 1, \dots$ be fixed such that $t_0 = 0, t_k < t_{k+1}, \lim_{k\to\infty} t_k = \infty$.

Definition 1 . *The function $g(t) : [0, \infty) \to \mathbb{R}$ is called a step function if $g(t) = g_k$ for $t_k \le t < t_{k+1}$ where $g_k = const, k = 0, 1, \dots$.*

Theorem 1 . *Let the following conditions be satisfied:*

1. The function $g(t) : [0, \infty) \to [0, \infty)$ is a step one such that $0 \le g_k \le t_k$ for $t \in [t_k, t_{k+1}), k = 0, 1, \dots$.

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2. The function $u(t) \in C([0,\infty),[0,\infty))$ satisfies the inequality

$$u(t) \le u_0 + \int_0^t p(s)u(s)\,ds + \sum_{0\le t_k<t} \beta_k u(g(t_k))$$

where $u_0 = u(0), \beta_k \ge 0$ for $k = 0,1,\dots$, $p(t) \in C([0,\infty),[0,\infty))$.

Then the inequality

$$u(t) \le u_0 \prod_{0\le t_k<t} \Big(1 + \beta_k e^{\int_0^{g_k} p(s)\,ds}\Big) e^{\int_0^t p(s)\,ds} \tag{1}$$

holds for $t \in [0,\infty)$.

P r o o f: Let $t \in [t_0, t_1)$. Since $0 \le g_0 \le t_0$, then $g_0 = 0$. From condition 2 of Theorem 1 it follows that the inequality

$$u(t) \le u_0 + \int_0^t p(s)u(s)\,ds + \beta_0 u(g(t_0)) = u_0(1+\beta_0) + \int_0^t p(s)u(s)\,ds \tag{2}$$

holds.

By the inequality of Gronwall-Bellman and the inequality (2) the following inequlity is fulfilled

$$u(t) \le u_0(1+\beta_0)e^{\int_0^t p(s)\,ds}.$$

The last inequality shows the validity of (1) for $t \in [t_0, t_1)$.

Suppose that the inequality (1) is satisfied for $t \in [t_{k-1}, t_k), k = 0,1,\dots,l$. We shall prove that the inequality (1) is fulfilled for $t \in [t_l, t_{l+1})$.

From condition 2 of Theorem 1 it follows that the function $u(t)$ satisfies the inequality

$$\begin{aligned} u(t) &\le u_0 + \int_0^t p(s)u(s)\,ds + \sum_{i=0}^{l} \beta_i u(g_i) \\ &= u_0 + \sum_{i=0}^{l-1} \int_{t_i}^{t_{i+1}} p(s)u(s)\,ds + \int_{t_l}^t p(s)u(s)\,ds + \sum_{i=0}^{l} \beta_i u(g_i)\ . \end{aligned}$$

By the continuity of the function $u(t)$ at the points $t_k, k = 0,1,\dots,l$ and the induction assumption we obtain

$$\begin{aligned} u(t) &\le u_0 + \sum_{i=0}^{l-1} \int_{t_i}^{t_{i+1}} p(s)u_0 \prod_{j=0}^{i} \Big(1+\beta_j e^{\int_0^{g_j} p(s)\,ds}\Big) e^{\int_0^s p(\tau)\,d\tau}\,ds \\ &+ \int_{t_l}^t p(s)u(s)\,ds + \sum_{i=0}^{l} \beta_i u_0 \prod_{j=0}^{i-1} \Big(1+\beta_j e^{\int_0^{g_j} p(s)\,ds}\Big) e^{\int_0^{g_i} p(s)\,ds} \\ &= u_0\Big[1 + \sum_{i=0}^{l-1} \prod_{j=0}^{i} \Big(1+\beta_j e^{\int_0^{g_j} p(s)\,ds}\Big)\Big(e^{\int_0^{t_{i+1}} p(s)\,ds} - e^{\int_0^{t_i} p(s)\,ds}\Big) \\ &+ \sum_{i=0}^{l} \prod_{j=0}^{i-1} \beta_i \Big(1+\beta_j e^{\int_0^{g_j} p(s)\,ds}\Big) e^{\int_0^{g_i} p(s)\,ds}\Big] + \int_{t_l}^t p(s)u(s)\,ds\ . \end{aligned}$$

From the last inequality the following inequality holds

$$u(t) \le u_0 \prod_{i=0}^{l} \Big(1 + \beta_i e^{\int_0^{g_i} p(s)\,ds}\Big) e^{\int_0^{t_l} p(s)\,ds} + \int_{t_l}^{t} p(s)u(s)\,ds \ . \tag{3}$$

By the inequality of Gronwall-Bellman and the inequality (3) the next inequality is fulfilled

$$\begin{aligned} u(t) \quad &\le \quad u_0 \prod_{i=0}^{l} \Big(1 + \beta_i e^{\int_0^{g_i} p(s)\,ds}\Big) e^{\int_0^{t_l} p(s)\,ds} e^{\int_{t_l}^{t} p(s)\,ds} \\ &= u_0 \prod_{i=0}^{l} \Big(1 + \beta_i e^{\int_0^{g_i} p(s)\,ds}\Big) e^{\int_0^{t} p(s)\,ds} \ . \end{aligned}$$

Therefore the inequality (1) is satisfied for $t \in [t_l, t_{l+1})$.

By induction of the intervals $[t_k, t_{k+1}), k = 0, 1, \dots$, we have obtained the validity of the inequality (1) for $t \in [0, \infty)$.

□

From Theorem 1 we obtain the following results:

Corollary 1 . *Let the conditions of Theorem 1 hold where $u_0 = 0$. Then $u(t) = 0$.*

□

Corollary 2 . *Let the following conditions be fulfilled:*

1. The function $g(t) : [0, \infty) \to [0, \infty)$ is a step one such that $0 \le g_k \le t_k$ for $t \in [t_k, t_{k+1}), k = 0, 1, \dots$.

2. M and N are positive constants.

3. The function $u(t) \in C([0, \infty), [0, \infty))$ satisfies the inequality

$$u(t) \le u_0 + \int_0^t Mu(s)\,ds + N \sum_{0 \le t_k < t} \triangle t_k u(g(t_k))$$

where $u_0 = u(0), \triangle t_k = t_{k+1} - t_k$ for $k = 0, 1, \dots$, for $t \in [0, \infty)$.

Then the inequality

$$u(t) \le u_0 \prod_{0 \le t_k < t} \Big(1 + N \triangle t_k e^{M g_k}\Big) e^{Mt} \tag{4}$$

holds for $t \in [0, \infty)$.

P r o o f: From Theorem 1 and setting $p(t) = M, \beta_k = N \triangle t_k, k = 0, 1, \dots$ for $t \in [0, \infty)$ we get

$$\begin{aligned} u(t) \quad &\le \quad u_0 \prod_{0 \le t_k < t} \Big(1 + N \triangle t_k e^{\int_0^{g_k} M\,ds}\Big) e^{\int_0^t M\,ds} \\ &= u_0 \prod_{0 \le t_k < t} \Big(1 + N \triangle t_k (e^{M g_k} - 1)\Big)(e^{Mt} - 1) \\ &\le u_0 \prod_{0 \le t_k < t} \Big(1 + N \triangle t_k e^{M g_k}\Big) e^{Mt}. \end{aligned}$$

□

3. Applications

Consider the initial value problem for the differential equation with a step function (IVP)

(5) $$x' = f(x(t), x(g(t))) \quad \text{for} \quad t \in J,$$

(6) $$x(0) = c_0$$

where $x \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $J \subseteq [0, \infty)$, c_0 is a constant, $g(t) : J \to J$ is a step function.

We denote by $PC^1(J, \mathbb{R})$ the set of all functions $u \in C(J, \mathbb{R})$ for which the derivative $u'(t)$ exists and it is piecewise continuous in J with points of discontinuity of first kind at the points $t_k,\ k = 1, 2 \ldots,\ u'(t_k) = u'(t_k + 0)$.

Definition 2 . *The function $x(t)$ is a solution of the IVP (5),(6) in the interval $J \subseteq [0, \infty)$ if the following conditions are fulfilled:*
1. $x(t) \in PC^1(J, \mathbb{R})$.
2. The function $x(t)$ turns the equalities (5),(6) into identities for $t \in J$.

Definition 3 . *The function $v(t) \in PC^1(J, R)$ is called a lower (upper) solution of the IVP (5), (6)in the interval $J \subseteq [0, \infty)$ if*

$$v'(t) \leq (\geq) f(v(t)), v(g(t))), \quad v(0) \leq (\geq) c_0.$$

Consider the initial value problem for the linear differential equation

(7) $$x'(t) = a(t)x(t) + b(g(t))x(g(t)) \text{ for } \ t \in [0, \infty), \quad x(0) = c_0$$

where $g(t) : [0, \infty) \to [0, \infty)$ is a step function, $a(t), b(t) : [0, \infty) \to \mathbb{R}, c_0$ is a constant.

Theorem 2 . *Let the following conditions be satisfied:*
1. The function $g(t) : [0, \infty) \to [0, \infty)$ is a step one such that $0 \leq g_k \leq t_k$ for $t \in [t_k, t_{k+1}), k = 0, 1, \ldots$.
2. The functions $a(t), b(t) \in C([0, \infty), \mathbb{R})$.
Then the IVP (7) has a unique solution for $t \in [0, \infty)$.

P r o o f: Let $t \in [t_0, t_1)$. Consider the IVP

(8) $$x'(t) = a(t)x(t) + b_0 s_0, \quad x(0) = c_0$$

where $s_0 = x(g_0) = x(0) = c_0, b_0 = b(g_0)$.

We denote $I(r, t) = \int_r^t a(\tau)\, d\tau$.

The solution of the IVP (8) exists for $t \geq 0$ and satisfies the equality

$$x_0(t) = c_0 e^{I(0,t)} (1 + b_0 \int_0^t e^{-I(0,\tau)}\, d\tau) \ .$$

Let $t \in [t_1, t_2)$. Consider the IVP

$$x'(t) = a(t)x(t) + b_1 s_1, \quad x(t_1) = c_1 \tag{9}$$

where $s_1 = x(g_1) = x_0(g_1)$, $c_1 = x_0(t_1)$, $b_1 = b(g_1)$.

The solution of the IVP (9) exists for $t \geq t_1$ and satisfies the equality

$$x_1(t) = e^{I(t_1,t)}\Big(x_0(t_1) + b_1 x_0(g_1) \int_{t_1}^{t} e^{-I(t_1,\tau)}\, d\tau\Big) .$$

Let $t \in [t_2, t_3)$. Consider the IVP

$$x'(t) = ax(t) + bs_2, \quad x(t_2) = c_2 \tag{10}$$

where $s_2 = x(g_2), c_2 = x_1(t_2)$. Since $g_2 \leq t_2$, then $s_2 = x_m(g_2)$ where

$$m = \begin{cases} 0 \text{ for} & g_2 \in [0, t_1] \\ 1 \text{ for} & g_2 \in (t_1, t_2] \end{cases} .$$

The solution of the IVP (10) exists for $t \geq t_2$ and satisfies the equality

$$x_2(t) = e^{I(t_2,t)}\Big(x_1(t_2) + b_2 x_m(g_2) \int_{t_2}^{t} e^{-I(t_2,\tau)}\, d\tau\Big) .$$

With the help of the solution $x_{k-1}(t)$ in the interval $[t_{k-1}, t_k)$ and the steps method, we construct the function $x_k(t)$ as a solution of the following IVP

$$x'(t) = a(t)x(t) + b_k s_k, \quad x(t_k) = c_k \text{ for } t \in [t_k, t_{k+1})$$

where $s_k = x_{k-i}(g_k), c_k = x_{k-1}(t_k)$ for $k = 1, 2, \ldots$ and some integer $i \leq k$.

Therefore

$$x_k(t) = e^{I(t_k,t)}\Big(x_{k-1}(t_k) + b_k x_{k-i}(g_k) \int_{t_k}^{t} e^{-I(t_k,\tau)}\, d\tau\Big) .$$

Define the function

$$x(t) = \begin{cases} x_0(t) \text{ for} & t \in [0, t_1) \\ x_1(t) \text{ for} & t \in [t_1, t_2) \\ & \ldots \\ x_k(t) \text{ for} & t \in [t_k, t_{k+1}] \\ & \ldots . \end{cases}$$

The function $x(t)$ is a solution of the IVP (7) in $[0, \infty)$.

We shall prove the uniqueness of the solution of the IVP (7). Suppose that $x_1(t)$, $x_2(t) \in PC^1([0, \infty), \mathbb{R})$ are two different solutions of the IVP (7). We set the function $\varphi = x_1(t) - x_2(t)$ and consider its absolute value

$$\begin{aligned} |\varphi(t)| \quad = \quad & \Big| c_0 + \int_0^t a(s)x_1(s)\, ds + \int_0^t b(g(s))x_1(g(s))\, ds - c_0 - \int_0^t a(s)x_2(s)\, ds \\ & - \int_0^t b(g(s))x_2(g(s))\, ds \Big| \leq \int_0^t |a(s)|\, |\varphi(s)|\, ds + \sum_{0 \leq t_k < t} |b(g_k)|\, |\varphi(g_k)| \triangle t_k . \end{aligned}$$

By Corollary 1 and the last inequality we obtain $\varphi(t) = 0$, i.e. $x_1(t) = x_2(t)$.

□

Theorem 3 . *Let the following conditions be satisfied:*

1. T, M, N are positive constants such that $t_p < T < t_{p+1}, p \in \mathbb{N} \cup \{0\}$ and $(M + N)T \leq 1$.

2. The function $g(t) : [0, T] \to [0, T]$ is a step one such that $0 \leq g_k \leq t_k$ for $t \in [t_k, t_{k+1}), k = 0, 1, \ldots, p$.

3. The function $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies the condition

$$|f(x_1, y_1) - f(x_2, y_2)| \leq M|x_1 - x_2| + N|y_1 - y_2| \ .$$

4. The functions $v(t), w(t) \in PC^1([0, T], \mathbb{R})$ are lower and upper solutions of the IVP (5),(6) and $v(t) \leq w(t)$ for $t \in [0, T]$.

Then the IVP (5), (6) has a unique solution for $t \in [0, T]$.

P r o o f: We note that conditions 1 and 4 of Theorem 3 are used to be proved the existence of the solution of the IVP (5), (6) by the monotone-iterative technique of Lakshmikantham in [4]. Here we shall prove only the uniqueness of the IVP (5), (6).

Suppose that $x_1(t), x_2(t) \in PC^1([0, T], \mathbb{R})$ are two different solutions of the IVP (5), (6). We set the function $\varphi = x_1(t) - x_2(t)$ and consider its absolute value

$$\begin{aligned} |\varphi(t)| &= \left| c_0 + \int_0^t f(x_1(s), x_1(g(s)))\, ds - c_0 - \int_0^t f(x_2(s), x_2(g(s)))\, ds \right| \\ &\leq \int_0^t |f(x_1(s), x_1(g(s))) - f(x_2(s), x_2(g(s)))|\, ds \ . \end{aligned}$$

From condition 3 of Theorem 3 the following inequality is fulfilled

$$|\varphi(t)| \leq \int_0^t M|\varphi(s)|\, ds + N \sum_{0 \leq t_k < t} |\varphi(g_k)| \triangle t_k \tag{11}$$

where $\triangle t_k = t_{k+1} - t_k, k = 0, 1, \ldots, p$.

By Corollary 2 and the inequality (11) it follows that $\varphi(t) = 0$, i.e. $x_1(t) = x_2(t)$. Therefore the assumption is not true.

□

Theorem 4 . *Let the conditions of Theorem 3 hold. Then the solutions of the differential equation (5) continuously depend on the initial condition for $t \in [0, T]$.*

P r o o f: Let $\varepsilon > 0$ be a fixed number.

Consider the differential equation (5) with an initial condition

$$x(0) = c_1 \tag{12}$$

and

$$x(0) = c_2 \tag{13}$$

Let $x_1(t), x_2(t) \in PC^1([0,T], \mathbb{R})$ be solutions of the IVP (5), (12) and the IVP (5), (13) respectively. Then the following equalities hold for $t \in [0,T]$

$$x_1(t) = c_1 + \int_0^t f(x_1(s), x_1(g(s)))\, ds$$

$$x_2(t) = c_2 + \int_0^t f(x_2(s), x_2(g(s)))\, ds\ .$$

Set the function $\varphi(t) = x_1(t) - x_2(t)$ and consider its absolute value. From conditions 1, 3 of Theorem 4 we get the inequality

$$|\varphi(t)| \leq |\varphi(0)| + \int_0^t M|\varphi(s)|\, ds + N \sum_{0 \leq t_k < t} |\varphi(g_k)| \triangle t_k \tag{14}$$

where $\triangle t_k = t_{k+1} - t_k, k = 0, 1, \ldots, p$.

Choose δ such that $\delta < \varepsilon \Big(\sum_{k=0}^{p} (1 + N \triangle t_k e^{M g_k}) e^{MT} \Big)^{-1}$.

If $|c_1 - c_2| = |\varphi(0)| < \delta$ for $t \in [0,T]$, then from the inequality (14) and Corollary 2 the following inequality holds

$$|\varphi(t)| < \delta \prod_{0 \leq t_k < t} \Big(1 + N \triangle t_k e^{M g_k}\Big) e^{Mt} \leq \delta \prod_{0 \leq t_k < T} \Big(1 + N \triangle t_k e^{M g_k}\Big) e^{MT} < \varepsilon$$

for $t \in [0,T]$.

Therefore the solutions of the differential equation (5) continuously depend on the initial condition for $t \in [0,T]$.

□

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