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MEASURABILITY OF SETS OF PAIRS OF INTERSECTING STRAIGHT LINES IN THE GALILEAN PLANE

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The measurable sets of pairs of intersecting straight lines and the corresponding invariant densities with respect to the group of the general similitudes and its subgroups are described.

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1. Introduction. In the affine version, the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y-axis of the basic affine coordinate system Oxy [7], [8], [10], [11]. The affine transformations leaving invariant the special direction Oy can be written in the form

$$\begin{aligned} x' &= a_1 + a_2 x, \\ y' &= a_3 + a_4 x + a_5 y, \end{aligned} \tag{1}$$

where $a_1, \ldots, a_5 \in \mathbb{R}$ and $a_2 a_5 \neq 0$.

It is easy to verify that the transformations (1) map a line segment and an angle of Γ_2 into a proportional line segment and a proportional angle with the coefficients of proportionality $|a_2|$ and $|a_2^{-1} a_5|$, respectively. Thus they form the group H_5 of the general similitudes of Γ_2. The infinitesimal operators of H_5 are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\,\frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = x\,\frac{\partial}{\partial y}, \quad X_5 = y\,\frac{\partial}{\partial y}.$$

In [1], [2] we proved the following results:

I. The four–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_4^1 = (X_1, X_2, X_3, X_4), \; H_4^2 = (X_1, X_2, X_3, X_5), \; H_4^3 = (X_2, X_3, X_4, X_5),$$
$$H_4^4 = (X_1, X_3, X_4, \alpha X_2 + X_5).$$

II. The three–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_3^1 = (X_1, X_2, X_3), \; H_3^2 = (X_1, X_2, X_5), \; H_3^3 = (X_1, X_3, X_4), \; H_3^4 = (X_2, X_3, X_4),$$

$$H_3^5=\big(X_2,\,X_3,\,X_5\big),\ H_3^6=\big(X_2,\,X_4,\,X_5\big),\ H_3^7=\big(X_1,\,X_3,\alpha\,X_2+\beta X_4+X_5\big),$$
$$H_3^8=\big(X_3,\,X_4,\,\alpha X_1+X_5\big),\ H_3^9=\big(X_3,\,X_4,\alpha\,X_2+X_5|\alpha\neq 0\big),$$
$$H_3^{10}=\big(X_3,\,X_2+2X_5,\,\alpha X_1+X_4|\alpha\neq 0\big).$$

III. The two–parametric subgroups of H_5 can be reduced to one of the following subgroups:

$$H_2^1=\big(X_1,\,X_2\big),\ H_2^2=\big(X_2,\,X_3\big),\ H_2^3=\big(X_2,\,X_4\big),\ H_2^4=\big(X_2,\,X_5\big),$$
$$H_2^5=\big(X_1,\,\alpha X_2+X_3\big),\ H_2^6=\big(X_1,\,\alpha X_2+X_5\big),\ H_2^7=\big(X_3,\,\alpha X_1+X_4|\alpha\neq 0\big),$$
$$H_2^8=\big(X_3,\,\alpha X_1+X_5\big),\ H_2^9=\big(X_3,\,\alpha X_2+\beta X_4+X_5|\alpha\neq 0\big),\ H_2^{10}=\big(X_4,\,\alpha X_2+X_3\big),$$
$$H_2^{11}=\big(X_4,\,\alpha X_2+X_5\big),\ H_2^{12}=\big(X_2+2X_5,\,\alpha X_1+X_4|\alpha\neq 0\big).$$

IV. The one–parametric subgroups of H_5 can be reduced to one of the following subgroups

$$H_1^1=\big(X_1\big),\ H_1^2=\big(X_2\big),\ H_1^3=\big(X_3\big),\ H_1^4=\big(X_4\big),\ H_1^5=\big(X_5\big),$$
$$H_1^6=\big(\alpha X_1+X_4|\alpha\neq 0\big),\ H_1^7=\big(X_1+X_5\big),\ H_1^8=\big(\alpha X_2+X_3|\alpha\neq 0\big),$$
$$H_1^9=\big(\alpha X_2+X_5|\alpha\neq 0\big),\ H_1^{10}=\big(\alpha X_2+\beta X_4+X_5|\alpha\beta\neq 0\big).$$

Here and everywhere in the text α and β are real constants.

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [9], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6], we find the measurable sets of pairs of intersecting straight lines in Γ_2 with respect to H_5 and its subgroups.

2. Measurability with respect H_5. Let $G_i:\ y=k_ix+n_i,\ \ i=1,2$, be two intersecting straight lines in Γ_2, i.e.

$$k_1k_2(k_2-k_1)\neq 0. \tag{2}$$

Under the action of (1) the pair $(G_1,\,G_2)(k_1,\,n_1,\,k_2,\,n_2)$ is transformed into the pair $(G_1',\,G_2')(k_1',\,n_1',\,k_2',\,n_2')$ as

$$\begin{aligned} &k_i'=a_2^{-1}(a_4+a_5k_i),\\ &n_i'=a_2^{-1}(a_2a_3-a_1a_4-a_1a_5k_i+a_2a_5n_i),\\ &a_2a_5\neq 0,\quad i=1,2. \end{aligned} \tag{3}$$

The transformations (3) form the so–called associated group $\overline{H_5}$ of H_5 [9; p.34]. The associated group $\overline{H_5}$ is isomorphic to H_5 and the invariant density with respect to H_5 of the pairs (G_1,G_2), if it exists, coincides with the invariant density with respect to $\overline{H_5}$ of the points $(k_1,\,n_1,\,k_2,\,n_2)$ in the set of parameters [9; p.33]. The infinitestimal operators of $\overline{H_5}$ are

$$Y_1=k_1\,\frac{\partial}{\partial n_1}+k_2\,\frac{\partial}{\partial n_2}\,,\quad Y_2=k_1\,\frac{\partial}{\partial k_1}+k_2\,\frac{\partial}{\partial k_2}\,,$$

$$Y_3 = \frac{\partial}{\partial n_1} + \frac{\partial}{\partial n_2}\,, \quad Y_4 = \frac{\partial}{\partial k_1} + \frac{\partial}{\partial k_2}\,,$$
$$Y_5 = k_1 \frac{\partial}{\partial k_1} + n_1 \frac{\partial}{\partial n_1} + k_2 \frac{\partial}{\partial k_2} + n_2 \frac{\partial}{\partial n_2}\,.$$

From (2) it follows that the infinitesimal operators Y_1, Y_2, Y_3 and Y_4 are arcwise unconnected. On the other hand, we obtain

$$Y_5 = -\frac{n_2 - n_1}{k_2 - k_1} Y_1 - Y_2 + \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1} Y_3$$

and since

$$Y_1\Big(-\frac{n_2 - n_1}{k_2 - k_1}\Big) + Y_2(-1) + Y_3\Big(\frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\Big) \neq 0,$$

we can state the following:

Theorem 1. *The sets of pairs of intersecting straight lines are not measurable with respect to the group $H5$ of the general similitudes and have not measurable subsets.*

3. Measurability with the respect to the subgroups of H_5.

The group $\overline{H_4^1} = (Y_1, Y_2, Y_3, Y_4)$, corresponding to the subgroup $H_4^1 = (X_1, X_2, X_3, X_4)$, is a simply transitive group and therefore it is measurable. The integral invariant function, [9; p.9] $f = f(k_1, n_1, k_2, n_2)$, satisfying the system of R. Deltheil [3; p.28], [9; p.ll]

$$Y_1(f) = 0, \quad Y_2(f) + 2f = 0, \quad Y_3(f) = 0, \quad Y_4(f) = 0$$

has the form

$$f = \frac{c}{(k_2 - k_1)^2}\,,$$

where $c = const \neq 0$. Thus we establish:

Theorem 2. *The pairs (G_1, G_2) of intersecting straight lines $G_i : y = k_i x + n_i$, $i = 1, 2$, have the invariant with respect to H_4^1 density*

$$d(G_1, G_2) = \frac{1}{(k_2 - k_1)^2}\, dG_1 \wedge dG_2, \tag{4}$$

where $dG_i = dk_i \wedge dn_i$, $i = 1, 2$, denotes the metric density for the straight lines in Γ_2.

Remark 1. Note that the (unoriented) angle between G_1 and G_2 is defined by the quantity

$$\Delta k = |k_2 - k_1|$$

and then (4) can be written in the form

$$d(G_1, G_2) = \frac{1}{(\Delta k)^2}\, dG_1 \wedge dG_2\,.$$

By arguments similar to the ones used above we examine the measurability of the set of set of pairs of intersecting straight lines with respect to all the rest subgroups of H_5. We collect the results in the following table:

subgroup	measurable set/subset	expression of the density
1	2	3
H_4^1		$(k_2-k_1)^{-2}dG_1\wedge dG_2$
H_4^2	it is not measurable and has not measurable subsets	
H_4^3	$n_2-n_1\neq 0$	$(k_2-k_1)^{-2}(n_2-n_1)^{-2}dG_1\wedge dG_2$
H_4^4 $\alpha\neq 1$		$(k_2-k_1)^{\frac{2(\alpha-2)}{1-\alpha}}dG_1\wedge dG_2$
H_4^4 $\alpha=1$	it is not measurable and has not measurable subsets	
H_3^1	$k_2=\lambda k_1,\ \ \lambda\neq 1,\ \ \ k_1\neq 0$	$\lvert k_1\rvert^{-1}dG_1\wedge dn_2$
H_3^2	$k_2=\lambda k_1,\ \ \lambda\neq 1,\ \ \ k_1(n_2-\lambda n_1)\neq 0$	$\lvert k_1\rvert^{-1}(n_2-\lambda n_1)^{-2}dG_1\wedge dn_2$
H_3^3	$k_2=k_1+\lambda,\ \ \lambda\neq 0$	$dG_1\wedge dn_2$
H_3^4	$n_2=n_1+\lambda$	$(k_2-k_1)^{-2}dG_1\wedge dk_2$
H_3^5	$k_2=\lambda k_1,\ \lambda\neq 1,\ k_1(n_2-n_1)\neq 0$	$\lvert k_1\rvert^{-1}(n_2-n_1)^{-2}dG_1\wedge dn_2$
H_3^6	$n_2=\lambda n_1,\ \ n_1\neq 0$	$\lvert n_1\rvert^{-1}(k_2-k_1)^{-2}dG_1\wedge dk_2$
H_3^7 $\alpha\neq 1$	$k_2=\frac{1}{1-\alpha}\{\lambda[(1-\alpha)k_1+\beta]-\beta\}$, $\lambda\neq 1$, $k_1\neq 0$	$\lvert k_1\rvert^{\frac{\alpha-3}{1-\alpha}}dG_1\wedge dn_2$
H_3^7 $\alpha=1$ $\beta=0$	it is not measurable and has not measurable subsets	
H_3^7 $\alpha=1$ $\beta\neq 0$	$k_2=k_1+\lambda,\ \ \ \lambda\neq 0$	$e^{\frac{-2k_1}{\beta}}dG_1\wedge dn_2$
H_3^8	$n_2=n_1+\frac{\alpha}{k_2-k_1}+\lambda(k_2-k_1)$	$\lvert k_2-k_1\rvert^{-3}dG_1\wedge dk_2$
H_3^9 $\alpha\neq 1$	$n_2=n_1+(k_2-k_1)^{\frac{1}{1-\alpha}}$	$\lvert k_2-k_1\rvert^{\frac{2\alpha-3}{1-\alpha}}dG_1\wedge dk_2$

1	2	3
H_3^9 $\alpha = 1$	$k_2 = k_1 + \lambda, \quad \lambda(n_2 - n_1) \neq 0$	$(n_2 - n_1)^{-2} dG_1 \wedge dn_2$
H_3^{10}	$n_2 = n_1 - \frac{1}{\lambda}(k_2 - k_1)^2, \quad \lambda \neq 0$	$\left(\frac{2-\alpha k_1}{\alpha(k_2-k_1)}\right)^2 dG_1 \wedge dk_2$
H_2^1	$k_1 \neq 0, \quad k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1$ $n_2 = \lambda_1 n_1 - \frac{1}{k_1}\lambda_2$	$\lvert k_1\rvert^{-1} dG_1$
H_2^2	$k_1 \neq 0, \quad k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1$ $n_2 = n_1 + \lambda_2$	$\lvert k_1\rvert^{-1} dG_1$
H_2^3	$n_1 = \lambda_1, \quad n_2 = \lambda_2$	$(k_2 - k_1)^{-2} dk_1 \wedge dk_2$
H_2^4	$k_1 n_1 \neq 0, \ k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1$ $n_2 = \lambda_2 n_1$	$\lvert k_1 n_1\rvert^{-1} dG_1$
H_2^5 $\alpha \neq 0$	$k_1 \neq 0, \quad k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1,$ $n_2 = (\lambda_1 - 1)\left(\frac{1}{\alpha}\ln\lvert k_1\rvert + n_1\right) + n_1 + \lambda_2$	$\lvert k_1\rvert^{-1} dG_1$
H_2^5 $\alpha = 0$	$k_1 = \lambda_1, \quad k_2 = \lambda_2, \quad \lambda_1 \neq \lambda_2$	$dn_1 \wedge dn_2$
H_2^6 $\alpha \neq 1$	$k_1 \neq 0, \quad k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1,$ $n_2 = \lambda_1 n_1 + \lambda_2 k_1^{\frac{1}{1-\alpha}}$	$\lvert k\rvert^{\frac{\alpha-2}{1-\alpha}} dG_1$
H_2^6 $\alpha = 1$	$k_1 = \lambda_1, \quad k_2 = \lambda_2, \quad \lambda_1 \neq \lambda_2$ $\lambda_1 n_2 - \lambda_2 n_1 \neq 0$	$(\lambda_1 n_2 - \lambda_2 n_1)^{-2} dn_1 \wedge dn_2$
H_2^7	$k_2 = k_1 + \lambda_1, \quad \lambda_1 \neq 0$ $n_2 = n_1 + \alpha(\lambda_1 k_1 + \frac{1}{2}\lambda_1^2) + \lambda_2$	dG_1
H_2^8	$\lambda_1 k_1 \neq 0, \quad k_2 = \lambda_1 k_1, \quad \lambda_1 \neq 1,$ $n_2 = n_1 + \alpha k_1 \ln\left\lvert\frac{k_1}{(\lambda_1 k_1)^{\lambda_1}}\right\rvert - \lambda_2 k_1$	$\lvert k_1\rvert^{-2} dG_1$
H_2^9 $\alpha \neq 1$	$k_2 = \frac{1}{1-\alpha}\left\{\lambda[(1-\alpha)k_1 + \beta] - \beta\right\},$ $\lambda \neq 1, \ n_2 = n_1 + \lambda_2\left[(1-\alpha)k_1 + \beta\right]^{\frac{1}{1-\alpha}}$	$\lvert k_1\rvert^{\frac{\alpha-2}{1-\alpha}} dG_1$
H_2^9 $\alpha = 1$ $\beta = 0$	$k_1 = \lambda_1, \quad k_2 = \lambda_2, \ \lambda_1 \neq \lambda_2,$	$(n_2 - n_1)^{-2} dn_1 \wedge dn_2$
H_2^9 $\alpha = 1$ $\beta \neq 0$	$k_2 = k_1 + \lambda, \quad \lambda \neq 0$ $n_2 = n_1 + \lambda_2 e^{\frac{1}{\beta}k_1}$	$e^{-\frac{1}{\beta}k_1} dG_1$
H_2^{10} $\alpha \neq 0$	$n_1 = -\frac{1}{\alpha}\ln\lvert k_2 - k_1\rvert + \lambda_1,$ $n_2 = -\frac{1}{\alpha}\ln\lvert k_2 - k_1\rvert + \lambda_2,$	$(k_2 - k_1)^{-2} dk_1 \wedge dk_2$
H_2^{10} $\alpha = 0$	$k_2 = k_1 + \lambda_1, \quad \lambda_1 \neq 0$ $n_2 = n_1 + \lambda_2,$	dG_1

1	2	3
H_2^{11} $\alpha \neq 1$	$n_1 = \lambda_1(k_2 - k_1)^{\frac{1}{1-\alpha}}$, $n_2 = \lambda_2(k_2 - k_1)^{\frac{1}{1-\alpha}}$	$(k_2 - k_1)^{-2} dk_1 \wedge dk_2$
H_2^{11} $\alpha = 1$	$k_2 = k_1 + \lambda_1, \quad \lambda_1 n_1 \neq 0$, $n_2 = \lambda_2 n_1$	$n_1^{-2} dG_1$
H_2^{12}	$n_1 = \lambda_1(k_2 - k_1)^2 - \frac{1}{2}\alpha k_1^2$, $n_2 = \lambda_2(k_2 - k_1)^2 - \frac{1}{2}\alpha k_2^2$,	$(k_2 - k_1)^{-2} dk_1 \wedge dk_2$
H_1^1	$k_1 = \lambda_1, \quad k_2 = \lambda_2, \quad \lambda_1(\lambda_1 - \lambda_2) \neq 0$ $n_2 = \frac{1}{\lambda_1}(\lambda_3 + \lambda_2 n_1)$	dn_1
H_1^2	$k_1 \neq 0, \quad n_1 = \lambda_1, \quad k_2 = \lambda_2 k_1$ $\lambda_2 \neq 1,\ n_2 = \lambda_3$	$\|k_1\|^{-1} dk_1$
H_1^3	$k_1 = \lambda_1, \quad k_2 = \lambda_2, \quad \lambda_1 \neq \lambda_2$ $n_2 = n_1 + \lambda_3$	dn_1
H_1^4	$n_1 = \lambda_1, \quad k_2 = k_1 + \lambda_2, \quad \lambda_2 \neq 0$ $n_2 = \lambda_3$	dk_1
H_1^5	$k_1 \neq 0, \quad n_2 = \lambda_1 k_1, \quad k_2 = \lambda_2 k_1$ $\lambda_2 \neq 1, \quad n_2 = \lambda_3 k_1$	$\|k_1\|^{-1} dk_1$
H_1^6	$n_1 = \frac{1}{2}\alpha k_1^2 + \lambda_1, \quad k_2 = k_1 + \lambda_2$, $\lambda_2 \neq 0,\ n_2 = \frac{1}{2}\alpha(k_1 + \lambda_2)^2 + \lambda_3$	dk_1
H_1^7	$k_1 \neq 0, \quad n_1 = k_1(\lambda_1 - \ln\|k_1\|)$, $k_2 = \lambda_2 k_1, \quad \lambda_2 \neq 1, \quad \lambda_2 \neq 0$, $n_2 = \lambda_2 k_1(\lambda_3 - \ln\|\lambda_2 k_1\|)$	$\|k_1\|^{-1} dk_1$
H_1^8	$k_1 \neq 0, \quad n_1 = -\frac{1}{\alpha}\ln\|k_1\| + \lambda_1$, $k_2 = \lambda_2 k_1, \quad \lambda_2 \neq 1$, $n_2 = -\frac{1}{\alpha}\ln\|k_1\| + \lambda_3$	$\|k_1\|^{-1} dk_1$
H_1^9 $\alpha \neq 1$	$k_1 \neq 0, \quad n_1 = \lambda_1 k_1^{\frac{1}{1-\alpha}}$, $k_2 = \lambda_2 k_1, \quad \lambda_2 \neq 1, \quad n_2 = \lambda_3 k_1^{\frac{1}{1-\alpha}}$	$\|k_1\|^{-1} dk_1$
H_1^9 $\alpha = 1$	$k_1 = \lambda_1, \quad k_2 = \lambda_2$, $(\lambda_1 - \lambda_2) n_1 \neq 0, \quad n_2 = \lambda_3 n_1$	$\|n_1\|^{-1} dn_1$
H_1^{10} $\alpha \neq 1$	$n_1 = \lambda_1\big[(1-\alpha)k_1 + \beta\big]^{\frac{1}{1-\alpha}}$, $k_2 = \frac{1}{1-\alpha}\big\{\lambda_2\big[(1-\alpha)k_1 + \beta\big] - \beta\big\}$, $\lambda_2 \neq 1, \quad n_2 = \lambda_3\big[(1-\alpha)k_1 + \beta\big]^{\frac{1}{1-\alpha}}$, $(1-\alpha)k_1 + \beta \neq 0$	$\|(1-\alpha)k_1 + \beta\|^{-1} dk_1$
H_1^{10} $\alpha = 1$	$k_1 = \beta\ln\|n_1\| + \lambda_1$, $k_2 = \beta\ln\|n_1\| + \lambda_2$, $n_2 = \lambda_3 n_1, \quad (\lambda_1 - \lambda_2) n_1 \neq 0$	$\|n_1\|^{-1} dn_1$

Remark. In the table λ, λ_1, λ_2, $\lambda_3 \in R$.

References

[1] A. V. Borisov. *On the subgroups of the similarity group in the Galilean plane.* C. R. Acad. Bulg. Sci., 46 (1993), no. 5, 19–21.

[2] A. V. Borisov. *Subgroups of the group of the general similitudes in the Galilean plane.* Math. Balkanica, 13 (1999), no 1–2, 55–84.

[3] R. Deltheil. *Probabilités Géométriques.* Gauthier–Villars, Paris, 1926.

[4] G.I. Drinfel'd. *On the measure of the Lie groups.* Zap. Mat. Otdel. Fiz. Mat. Fak. Kharkov. Mat. Obsc., 21 (1949), 47–57 (Russian).

[5] G.I. Drinfel'd, and A.V. Lucenko. *On the measure of sets of geometric elements.* Vest. Kharkov. Univ., 31 (1964), no. 3, 34–41 (Russian).

[6] A.V. Lucenko. *On the measure of sets of geometric elements and their subsets.* Ukrain. Geom. Sb., 1 (1965), 39–57 (Russian).

[7] N.M. Makarova. *Galilean–Newtonian geometry I–III.* Uč. Zap. Orehovo-Zuev. Ped. Inst., 1 (1955), 83–95; 7 (1957), 7–27; 7 (1957), 29–59 (Russian).

[8] H. Sachs. *Ebene Isotrope Geometrie.* Vieweg, Braunschweig/Wiesbaden, 1987.

[9] M.I. Stoka. *Geometrie Integrala.* Ed. Acad. RPR, Bucuresti (1967).

[10] K. Strubecker. *Geometrie in einer isotropen Ebene I–III.* Math. Naturwiss. Unterricht, 15 (1962/1963), no. 7, 297–306; no. 8, 343–351; no. 9, 385–394.

[11] I.M. Yaglom. *A Simple Non-Euclidean Geometry and its Physical Basic.* Springer, Berlin, 1979.

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