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MÖBIUS PLANES WITH SHARPLY 3-TRANSITIVE GROUP OF AFFINE PROJECTIVITIES

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Abstract

In this paper we consider a class of Möbius planes, the so called (F)-planes. We prove that an (F)-plane with sharply 3-transitive group of affine projectivities is determined by the set of all circles through a fixed point and one further circle.

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Let $(P, \mathfrak{K})$ be a Möbius plane with point set P and circle set $\mathfrak{K}$ [1]. For each point $p \in P$ we define $\mathfrak{K}(p) := \{C \in \mathfrak{K} \mid p \in C\}$ and $\mathfrak{K}^p := \{C \setminus \{p\} \mid C \in \mathfrak{K}(p)\}$. Then $\mathcal{A}(p) := (P \setminus \{p\}, \mathfrak{K}^p)$ is an affine plane.[2] $\mathcal{A}(p)$ is called the *affine derivation of* $(P, \mathfrak{K})$ *at the point* p.

There are different types of perspectivities to define in Möbius planes (cf. [2]). In this paper we are concerned with affine perspectivities. Let $A, B \in \mathfrak{K}(p)$. A mapping $\pi : A \to B$ is called *affine perspectivity* with base point p, if $\pi(p) = p$ and if the restriction of π onto $A \setminus \{p\}$ is a parallel perspectivity in the affine derivation $\mathcal{A}(p)$. Now let A, B be two arbitrary circles and $\varrho : A \to B$ a mapping. ϱ is called *affine projectivity* if there exists a finite number of affine perspectivities $\pi_1, \dots \pi_n$ such that $\varrho = \pi_1 \circ \dots \circ \pi_n$. For any circle $C \in \mathfrak{K}$ we denote by Γ^C the group of all affine projectivities from C onto C. Since for $A, C \in \mathfrak{K}$ the groups Γ^A and Γ^C are isomorphic we may write Γ instead of Γ^C and may call $\Gamma =: \Gamma(P, \mathfrak{K})$ the *group of affine projectivities* of the Möbius plane $(P, \mathfrak{K})$.

The following results can be found in [4]:

(1) The group Γ acts 3-transitively on C.

[1] For the definition see [1], [3].
[2] This property is characterizing Möbius planes.

(2) If the Möbius plane $(P, \mathfrak{K})$ is Miquelian then Γ is sharply 3-transitive.

The theorem, that for any projective plane the group of projectivities of a line is sharply 3-transitive if and only if the plane is Pappian, gives rise to the question wether the converse of (2) holds. A first step in this direction is the following result [4, (3.1)]:

(3) If Γ is sharply 3-transitive then each affine derivation of $(P, \mathfrak{K})$ is Pappian.

This paper deals with (F)-planes with sharply 3-transitive group Γ. A Möbius plane is called *(F)-plane* if each circle which is tangent to three circles of a touching pencil belongs to this pencil. As corollaries of the following theorem we get two further results concerning the above question.

Theorem 1 *Let $(P, \mathfrak{K})$ be an (F)-plane of order greater than 3 and $\mathfrak{K}'$ a second subset of the power set of P such that $(P, \mathfrak{K}')$ is a Möbius plane. Let w be a point such that $\mathfrak{K}(w) = \mathfrak{K}'(w)$ and $\mathfrak{K} \cap \mathfrak{K}' \neq \mathfrak{K}(w)$. If Γ and $\Gamma' := \Gamma(P, \mathfrak{K}')$ are sharply 3-transitive then $\mathfrak{K} = \mathfrak{K}'$.*

Corollary 1 *Let $(P, \mathfrak{K})$ be an (F)-plane with sharply 3-transitive group Γ. Let w be a point and S a circle not containing w such that S is a conic of the affine derivation $\mathcal{A}(w)$, then $(P, \mathfrak{K})$ is Miquelian.*

Proof. Since S is a conic of $\mathcal{A}(w)$ there exists a subset $\mathfrak{K}'$ of the power set of P such that $(P, \mathfrak{K}')$ is a Miquelian Möbius plane, $S \in \mathfrak{K}'$ and $\mathfrak{K}(w) = \mathfrak{K}'(w)$. By (2) the group Γ' is sharply 3-transitive, and therefore $\mathfrak{K} = \mathfrak{K}'$ by the theorem 1.

Corollary 2 *Every finite Möbius plane $(P, \mathfrak{K})$ of odd order with sharply 3-transitive group Γ is Miquelian.*

Proof. The theorem of Qvist (cf. [5, p. 50]) implies that every finite Möbius plane is an (F)-plane. Let $w \in P$ and $S \in \mathfrak{K} \setminus \mathfrak{K}(w)$. By (3) the affine derivation $\mathcal{A}(w)$ is Pappian. Hence, by Segre [6], S is a conic of $\mathcal{A}(w)$, and we can apply corollary 1.[3]

To prove the theorem we need the following equivalence relations on the circle sets $\mathfrak{K}$ and $\mathfrak{K} \setminus \mathfrak{K}(w)$. Two circles A and B are called *equivalent* if there is a finite number of circles $C_1, \ldots, C_n$ such that $A = C_1, C_n = B$ and $|C_i \cap C_{i+1}| = 1$ for $i = 1, \ldots, n-1$. They are called *equivalent with respect to the point* w if in addition $w \notin C_i$ holds.

(4) Let $(P, \mathfrak{K})$ be a Möbius plane of order greater than 3, $w \in P$ and $A, B \in \mathfrak{K} \setminus \mathfrak{K}(w)$. If A is equivalent to B then A is equivalent to B with respect to w.

Proof. 1. Let the order of $(P, \mathfrak{K})$ be even. By the theorem of Qvist (cf. [5, p. 50]) there is exactly one point $a_w \in P \setminus A$, $a_w \neq w$ such that each circle through a_w and w is touching A. This point is called the *knot* of A with respect to w. Let b_w denote the knot of B with respect to w. There is a circle T through w, a_w, b_w. Let $p \in P \setminus (A \cup B \cup T)$

[3] For the last conclusion we could also refer to the theorem of J.A.Thas from [7], if the order is different from 11, 23, 59.

and a_p and b_p denote the knot of A and B with respect to w respectively. Then there is a circle C with $p, a_p, b_p \in C$. By the theorem of Qvist this circle C touches A and B and does not pass through w.

2. Let the order of $(P, \mathfrak{K})$ be odd. Let $X, Y, T \in \mathfrak{K}$ such that $w \notin X,\ w \in T$ and $|X \cap T| = 1$, $|Y \cap T| = 1$. Let $\{x\} := X \cap T$. By the theorem of Qvist there is a circle T' through x such that $T \cap T' = \{x\}$ and $|T' \cap Y| = 1$. It holds $w \notin T'$. Hence, in a sequence $C_1, \ldots, C_n$ of circles with $w \notin C_1, C_n$ and $|C_i \cap C_{i+1}| = 1$ every circle C_i with $w \in CI$ can be replaced by a circle C_i' with $w \notin C_i'$ and $|C_{i-1} \cap C_i'| = 1 = |C_i' \cap C_{i+1}$. Thus, —if A is equivalent to B, then A is also equivalent to B with respect to w.

Now we prove the theorem 1 in several steps. For each circle $X \in \mathfrak{K} \setminus \mathfrak{K}(w)$ we denote by $)X($ the class of all circles equivalent to X if $(P, \mathfrak{K})$ is of finite order, and the class of all circles equivalent to X with respect to w if $(P, \mathfrak{K})$ is of infinite order. Note that $)X(\subset \mathfrak{K}$ if the order is finite and $)X(\subset \mathfrak{K} \setminus \mathfrak{K}(w)$ if the order is infinite.

(a) Let $A \in \mathfrak{K} \setminus \mathfrak{K}(w)$ and $x, y \in P \setminus \{w\}$ two distinct points. Then there exist three circles $B_1, B_2, B_3 \in)A($ such that $B_1 \cap B_2 = B_2 \cap B_3 =_3 \cap B_1 = \{x, y\}$.

Proof. In case of finite order the theorem of Qvist (cf. [5, p. 50]) implies that there are at most two classes $)A($ and $)B($ and that half of the circles through two points belong to $)A($ [4], and we are done as the order is greater than 3.

Now we consider the case of infinite order. We may assume $x, y \notin A$. Let T denote the unique circle with $w, x, y \in T$. For $a \in A \setminus T$ let $X_a \in \mathfrak{K}(x) \setminus \mathfrak{K}(w)$ and $Y_a \in \mathfrak{K}(y) \setminus \mathfrak{K}(w)$ denote the unique circle with $X \cap A = \{a\}$ and $Y \cap A = \{a\}$. We have $w \notin X_a$ or $w \notin Y_a$ for otherwise $X_a = Y_a = T$ contradicting $a \notin T$. Since $(P, \mathfrak{K})$ is an (F)-plane there are at most two points $a', a'' \in A \setminus T,\ a', a'' \neq a$ such that $X_a \cap X_{a'} = \{x\}$, $Y_a \cap Y_{a''} = \{y\}$. Therefore, as the order of $(P, \mathfrak{K})$ is not bounded, we may assume that there are four circles $X_1, X_2, X_3, X_4 \in \mathfrak{K}(x)$ with $w, y \notin X_i$, $|A \cap X_i| = 1$ and $|X_i \cap X_j| = 2$ for $i \neq j$. Then at least three of the four distinct circles $B_1, B_2, B_3, B_4 \in \mathfrak{K}(y)$ with $B_i \cap X_i = \{x\}$ are equivalent to A with respect to w.

For $p \in P,\ A, B \in \mathfrak{K}(p)$ and $a \in A \setminus B,\ b \in B \setminus A$ we denote by $[p, A, B, a, b]$ the affine perspectivity in $(P, \mathfrak{K})$ from A onto B with base point p mapping a onto b. For $A', B' \in \mathfrak{K}'(p)$ and $a \in A' \setminus B',\ b \in B' \setminus A'$ the corresponding affine perspectivity in $(P, \mathfrak{K}')$ is denoted by $[p, A', B', a, b]'$.

(b) Let $A, B \in \mathfrak{K}$ with $w \notin A, B$ and $|A \cap B| = 1$. If $A \in \mathfrak{K}'$ then $B \in \mathfrak{K}'$.

Proof. Let $\{u\} := A \cap B,\ a \in A \setminus \{u\},\ G \in \mathfrak{K}$ with $w, a, u \in G$, let $\{u, b\} := B \cap G$ and $B' \in \mathfrak{K}'$ with $A \cap B' = \{u\},\ b \in B'$. Now let $x \in B \setminus \{u, b\}$. Let $H \in \mathfrak{K}$ with $w, x, u \in H$ and $\{u, h\} := A \cap H$. We have $G, H \in \mathfrak{K}'$. For $p \in \{w, u\}$ we define $\pi_p := [p, G, H, a, h]$ and $\pi_p' := [p, G, H, a, h]'$. Since $\mathfrak{K}(w) = \mathfrak{K}'(w)$ we have $\pi_w = \pi_w'$. Hence the identities $\pi_p(u) = u = \pi_p'(u)$, $\pi_p(w) = w = \pi_p'(w)$ and $\pi_p(a) = h = \pi_p'(a)$ together with the assumption on Γ and Γ' imply $\pi_u = \pi_w = \pi_w' = \pi_u'$ and consequently $x = \pi_u(b) = \pi_u'(b) \in B'$. In the same way we obtain $B' \subset B$, hence $B = B' \in \mathfrak{K}'$.

A direct consequence of (b) is because of $\mathfrak{K}(w) = \mathfrak{K}'(w)$ with (4) in mind

[4] If the order is even then there is only one class.

(c) If $S \in \mathfrak{K} \cap \mathfrak{K}' \setminus \mathfrak{K}(w)$ then $)S(\subset \mathfrak{K}'$ and $)S(=)S('$.

For $S \in \mathfrak{K} \cap \mathfrak{K}'$ and $s \in S$ we denote by Γ_s^S and $\Gamma_s'^S$ the stabilizer of s in Γ^S and Γ'^S respectively.

(d) If $S \in \mathfrak{K} \cap \mathfrak{K}' \setminus \mathfrak{K}(w)$ then there is an $s \in S$ such that $\Gamma_s^S = \Gamma_s'^S$.

Proof. For $g \in S$ there is a circle $G \in \mathfrak{K}(w)$ with $G \cap S = \{g\}$. Let $r \in S \setminus \{g\}$. By (a) there is a circle $R \in)S($ with $S \cap R = \{g, r\}$ and there is an $h \in G \setminus \{g\}$ with $R \cap G = \{g, h\}$. By (c) we have $)R(=)S(\subset \mathfrak{K}'$, thus $[g, G, S, h, r] = [g, G, S, h, r]' =: \phi$ and consequently $\Gamma_{\phi(w)}^S = \phi \Gamma_w^G \phi^{-1}$ and $\Gamma_{\phi(w)}'^S = \phi \Gamma_w'^G \phi^{-1}$. In every affine plane the group of parallel projectivities is 2-transitive. Hence the stabilizers Γ_w^G and $\Gamma_w'^G$ coincide since $\mathfrak{K}(w) = \mathfrak{K}'(w)$ and both Γ and Γ' is sharply 3-transitive. Thus for $s := \phi(w)$ we obtain $\Gamma_s^S = \phi \Gamma_w^G \phi^{-1} = \phi \Gamma_w'^G \phi^{-1} = \Gamma_s'^S$.

Now we are able to show

(e) $\mathfrak{K} = \mathfrak{K}'$

Proof. Let $S \in \mathfrak{K} \cap \mathfrak{K}' \setminus \mathfrak{K}(w)$. By (c) we have $)S(\subset \mathfrak{K}'$. Now assume there is an $A \in \mathfrak{K}$ with $A \notin)S($ and $w \notin A$. By (d) there is an $s \in S$ with $\Gamma_s^S = \Gamma_s'^S$. By (a) there exists $T \in)A($ with $s \in T$ and $w \notin T$. Since $T \notin)S($ there is a $t \in P$ with $t \neq s$ and $S \cap T = \{s, t\}$. Let $r \in T \setminus S$. By (a) there are $R_1, R_2 \in)S($ such that $R_1 \cap R_2 = \{r, s\}$ and $|R_i \cap S| = 2$. Furthermore let $T' \in \mathfrak{K}'$ be the uniquely determined circle in $(P, \mathfrak{K}')$ with $r, s, t \in T'$.

For $i = 1, 2$ we define $\{s, r_i\} := S \cap R_i$, $\pi_i := [s, S, T, r_i, r]$ and $\pi_i' := [s, S, T', r_i, r]'$. For $\pi := \pi_2^{-1} \pi_1$ and $\pi' := {\pi_2'}^{-1} \pi_1'$ we have $\pi(s) = s = \pi'(s)$, $\pi(t) = t = \pi'(t)$ and $\pi(r_1) = r_2 = \pi'(r_1)$. Hence $\pi = \pi'$ since $\Gamma_s^S = \Gamma_s'^S$.

For $x \in T$ and $i = 1, 2$ let $X_i \in \mathfrak{K}$ with $R_i \cap X_i = \{s\}$, $x \in X_i$ and $\{s, x_i\} := S \cap X_i$. Then $\pi'(x_1) = \pi(x_1) = x_2$. By (c) we have $X_i \in \mathfrak{K}'$ since $X_i \in)R(=)S($. Hence $x \in X_1 \cap X_2 \subset T'$ since $\pi'(x_1) = x_2$. Thus $T \subset T'$. In the same way we get $T' \subset T$.

Therefore $T = T' \in \mathfrak{K} \cap \mathfrak{K}' \setminus \mathfrak{K}(w)$ and $A \in \mathfrak{K}'$ by (c). Hence $\mathfrak{K} \subset \mathfrak{K}'$. Since both $(P, \mathfrak{K})$ and $(P, \mathfrak{K}')$ are Möbius planes $\mathfrak{K} \subset \mathfrak{K}'$ implies $\mathfrak{K} = \mathfrak{K}'$.

References

[1] W. Benz, *Vorlesungen über Geometrie der Algebren.* Springer, Berlin-Heidelberg-New York 1973.

[2] H. Karzel & H.-J. Kroll, *Perspectivities in circle geometries.* Geometry - von Staudt's point of view, Proc. NATO Adv. Study Inst., Bad Windsheim/Ger. 1980 (K. Strambach & P. Plaumann, eds.), 1981, pp. 51–99.

[3] H.-J. Kroll, *Anordnungsfragen in Benz-Ebenen.* Abh. Math. Semin. Univ. Hamb. **46** (1977), 217–255.

[4] H.-J. Kroll, *Perspektivitäten in Benz-Ebenen.* Beitr. geom. Algebra, Proc. Symp. Duisburg 1976, 1977, pp. 203–207.

[5] H. Lüneburg, *Die Suzukigruppen und ihre Geometrien.* Springer, Berlin 1965.

[6] B. Segre, *Ovals in finite projective planes.* Can. J. Math. (1955), 414–416.

[7] J. A. Thas, *Solution of a classical problem on finite inversive planes.* Finite geometries, buildings, and related topics. Pap. Conf., Pingree Park/CO (USA) 1988, 1990, pp. 145–159.

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