ЮБИЛЕЙНА НАУЧНА СЕСИЯ – 30 години ФМИ,
ПУ “Паисий Хилендарски”, Пловдив, 3-4.11.2000

CONSTRUCTING DIOPHANTIAN FIGURES: ALGORITHMS AND GEOMETRY

Maria Nikolova Brancheva

In this paper we expose different constructions of Diophantian figures obtained with the help of computer experiments and geometric considerations. The notion of Diophantian figure was introduced in [1] and developed in [2].

AMS Subject Classification:05C12, 11A99, 05B99

1. **INTRODUCTION.** Let us recall that the role of numbers in Antic Greek geometry was not of principal importance. Having the line and the compass as instrument one obtain the possibility to compare the segments without any information about their lengths. The possibility of geometric constructions with the help of the above mentioned instruments is based on the continual 2-dimensional structure of the plane.

After Decartes and Fermat, the continualistic understanding of the plane is expressed by real numbers. This is the analytic geometry which reduces the geometric constructions to a solving of algebraic equations and making different calculations.

We are interested of a kind of plane geometry in which only integer coordinates are admitted. This synthesis of analytic geometry with Greek arithmetic spirit give rise to a large use of computer algorithms and calculations. We hope this is a motivation for our work from the point of view of mathematical education.

2. **RECALL OF DEFINITIONS.** We shall consider the so-called Diophantian plane, i.e. the Cartesian product **Z x Z,** where by **Z** is denoted the ring of integers. Clearly, Diophantian plane is the lattice of all points in the plane of Decartes **R x R**, (**R** is the field of real numbers) with integer coordinates (x,y), $x \in \mathbf{Z}$, $y \in \mathbf{Z}$

We recall that *Diophantian figure* is defined by a set of points in Diophantian plane under the condition that the distance between each couple of its points is a positive integer. A Diophantian figure is called linear if its points lie on a line in the plane of Decartes. In the contrary, i.e. in the case the figure contains at least three different non-collinear points, we say that we have a flat Diophantian fugure.

According to a theorem of P.Erdos each Diophantian figure defined by an infinite number of different points is linear. So, flat Diophantian figures are always with a finite number of points.

3. **DIOPHANTIAN TRIANGLES.** Diophantian figures admit triangulation with Diophantian triangles [2], called Diophantian triangulation. This implies that these figures can be constructed by Diophantian triangles.

A large class of Diophantian figures can be obtained from Pythagorean triangles with common cathetus (leg).

PROPOSITION 1. The set of all Pythagorean triangles with fixed common cathetus is finite.

Proof. Let (**a**, y, z) be an arbitrary Pythagorean triple with fixed cathetus **a**. Here by z is denoted the hypotenuse and by y the other cathetus

We shall consider the equality

$$\mathbf{a}^2 = z^2 - y^2 = (z - y)(z + y).$$

Clearly z - y and z + y must take a finite numbers of integer values, as they are divisors of $\mathbf{a}^2$. Indeed, if $z - y = \mathbf{p}_k$, and $z + y = \mathbf{q}_l$, $\mathbf{a}^2 = \mathbf{p}_k\mathbf{q}_l$ we obtain

$$z = 1/2(\mathbf{p}_k + \mathbf{q}_l) \quad \text{and} \quad y = 1/2(\mathbf{p}_k - \mathbf{q}_l),$$

where k and l are indices which take a finite numbers of values. □

REMARK. Proposition 1 follows and by Erdos theorem cited above.

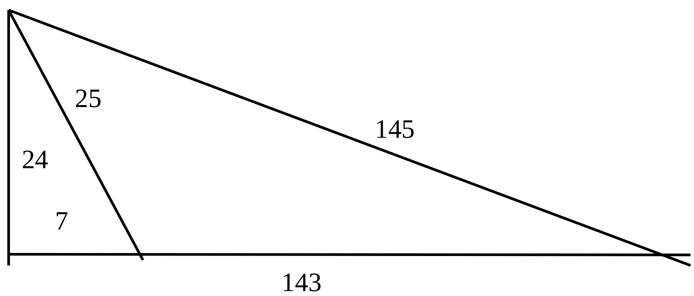
CONSEQUENCE. Each system of Pythagorean triangles with common cathetus determines a Diophantian figure. The examples below are obtained by computer program selecting Pythagorean triples with common cathetus.

(**24**, 143, 145), (**24**, 7, 25),

(**660**,12091,121090), (**660**, 4331, 4381), (**660**, 2989, 3061),
(**660**, 989, 1189), (**660**, 779, 1021), (**660**, 259, 709).

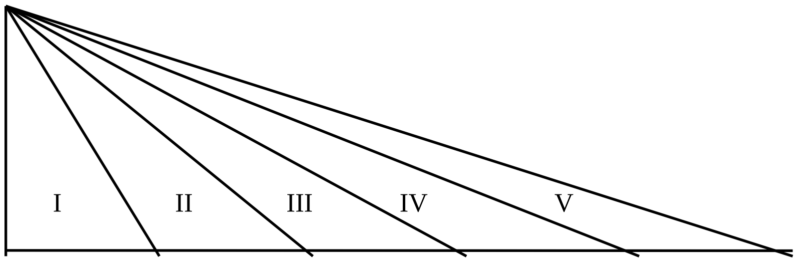
(**840**, 19591, 19609), (**840**, 11009, 11041), (**840**, 7031, 7081),
(840, 3551, 3649), **(840**, 1081, 1369), **(840,** 559, 1009),
(840, 41, 841).

We give a sketch for the first example. It defines a Diophantian figure with 4 points (vertexes)



In the case of second example, a Diophantian figure with 7 vertexes is defined, in the third case - with 8 vertexes.

In all cases a triangulation composed by one Pythagorean triangle and other Diophantian ones (respectively 1, 6 and 7) is determined.



A mathematical algorithm can be developed as follows. Setting

$$\boldsymbol{a} = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} \quad \text{i.e.} \quad \boldsymbol{a}^2 = a_1^{2\alpha_1} a_2^{2\alpha_2} \dots a_n^{2\alpha_n}$$

we can compare the divisors of $\mathbf{a}^2$ with divisors of the product (z - y)(z + y). As a result we obtain a number of equations of the following form

$$z - y = \dots.. \qquad \text{and} \qquad z + y = \dots$$

Solving these equations we obtain the result.

The number of divisors of $\mathbf{a}^2$ is equal to $(2\alpha_1 + 1)(2\alpha_2 + 1) \dots (2\alpha_n + 1)$, which shows that the number of possible systems for z and y is very big, when α_k are great enough. However we take the following example: $\mathbf{a} = 24$, $\mathbf{a}^2 = 576 = 2^6 3^2$ Here we have (2.3 + 1)(2 + 1) = 21 divisors, and respectively a list of system for z and y as above. For instance

$z - y = 1$, $z + y = 2^6 3^2$ which gives non-integer solutions, and many other systems which do not give integer solutions, but we have

$z - y = 2.3^2$, $z + y = 2^5$ which gives $z = 25$, $y = 7$,

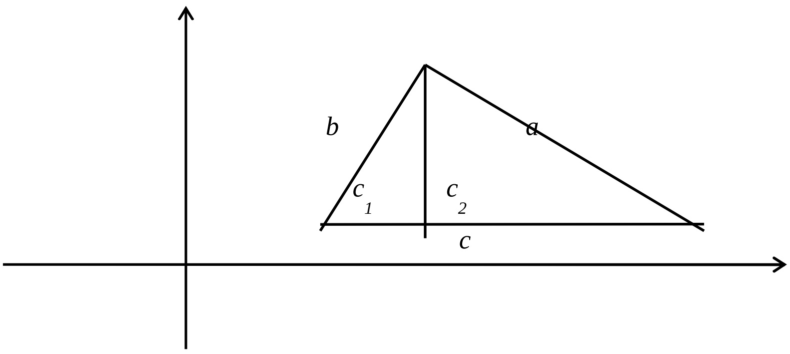
$z - y = 2$, $z + y = 2^5 3^2$ which give $z = 145$, $y = 143$. □

PROPOSITION 2. If (a, b, c) are lengths of the sides of a Diophantian triangle, then $a^2 + b^2 + c^2$ is an even number.

Proof. In the case of Pythagorean triangle we have

$$a^2 + b^2 + c^2 = 2c^2$$

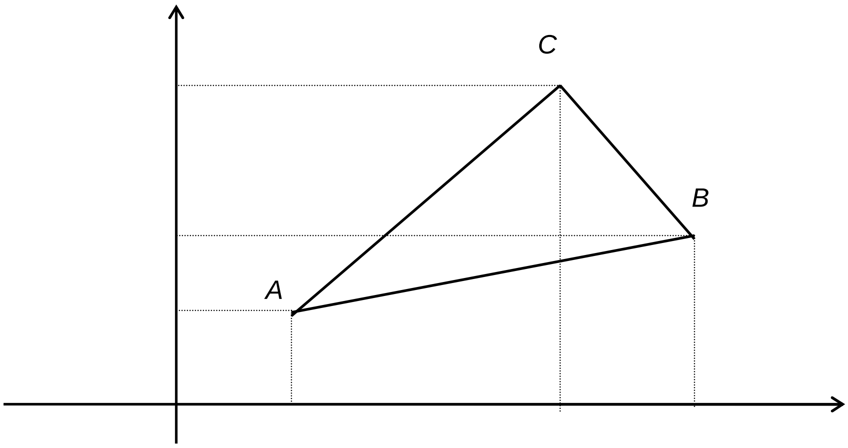
In the case of a horizontal side



we have

$$a^2 + b^2 + c^2 = 2h^2 + 2c^2 - 2c_1c_2 .$$

In the general case of Diophatian triangle ABC with coordinates of the vertexes $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$



we have

$$a^2 + b^2 + c^2 = 2(a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2) -$$
$$2(a_1b_1 + a_1c_1 + b_1c_1) - 2(a_2b_2 + a_2c_2 + b_2c_2) \square$$

CONSEQUENCE. In each Diophantian triangle there are always an even number of sides with odd length, i.e. 0 or 2. The distribution of the parity for the sides in a Diophantian figure with fixed Diophantian triangulation can be reconstructed starting by an arbitrary component of the considered triangulation.

4. BIG PYTHAGOREAN TRIANGLES: A HYPOTESE.

PROPOSITION 3. There exist Pythagorean triangles with an arbitrary great length of its sides.

Proof. Let a and b be an arbitrary couple of fixed natural numbers. We shall consider right triangles with $a + x$ and b as legs. Considering x and y as unknowns we introduce the following equation

$$(a + x)^2 + b^2 = y^2 .$$

This is an equation of a curve of second degree with integer coefficients. It has a finite number of solutions (x,y) with integer coordinates. Let (x_0, y_0) be such solution. Then the triple $(a + x_0, b, y_0)$ is a Pythagorean triple. $\square$

Now we introduce the following integer-valued function $k = \pi(n)$, $n \in \mathbf{N}$.

By definition

$\pi(n) = 0$ if n is not a cathetus of Pythagorean triangle,

$\pi(n)$ is the number of all Pythagorean triangles with n as a cathetus.

According to Proposition 1 the above mentioned definition make sense

Here we state the following hypothesis: the function $k = \pi(n)$ is a slowly increasing function and it is true that

$$\lim_{n\to\infty} (\pi(n)/n) = 0 .$$

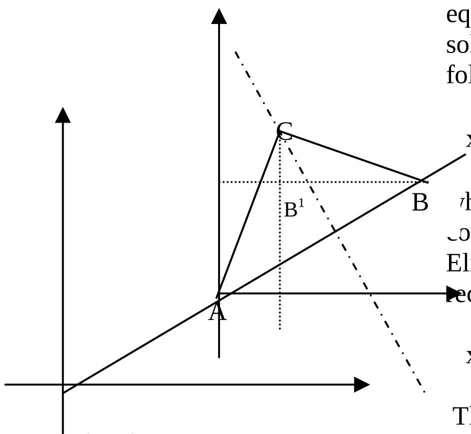
5. A DIOPHANTIAN EQUATION. Having in mind the condition of the PROPOSITION 2, here we can accept that $a_1 = a_2 = 0$.

PROPOSITION 4. Setting $c_1 = x_1$, $c_2 = x_2$, we obtain the following Diophantian equation of first degree for x_1 and x_2

$$2b_1x_1 + 2b_2x_2 = c^2 + b^2 - a^2$$

For given coordinates (b_1, b_2) (i.e. for given position and length c of the segment AB) we have that the solutions of the obtained Diophantian equations, if they exist, determine points on a line, which is perpendicular to AB, the lengths a and b being fixed as its may.

Proof. Clearly, we have $c^2 = b_1^2 + b_2^2$ and $b^2 = x_1^2 + x_2^2$. On the other hand we have $a^2 = (b_1 - x_1)^2 + (b_2 - x_2)^2$ (See the triangle B^1BC on the draw below). The above mentioned three equalities implies the announced proposition.



According to the general theory of Diophantian equations of first degree, if (x_1^0, x_2^0) is one solution, then all solutions are given by the following formulas

$$x_1 = x_1^0 + b_2/\langle b_1,b_2\rangle\, t, \quad x_2 = x_2^0 - b_1/\langle b_1,b_2\rangle\, t,$$

where t is an integer, and $\langle b_1,b_2\rangle$ is the gread common divisor of b_1 and b_2. [3] Eliminating t from the above two equations we receive for the solutions (x_1, x_2) that

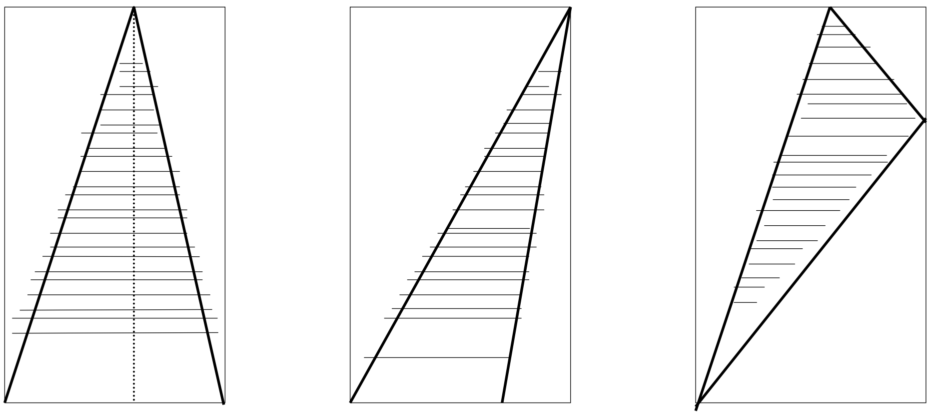
$$x_2 - x_2^0 = -b_1/b_2(x_1 - x_1^0) .$$

This is an equation of a line, which pass by the point (x_1^0, x_2^0), perpendicularly to the line of the segment AB. □

REMARK. This proposition concerns the construction of a Diophantian triangle with given sides a, b, c and given coordinates (b_1, b_2). The analogous construction in the classical planimetry is well known. Our construction don't follows from the classical one. It is of arithmetical character. Of course, the number of solutions is not infinite, when c, a, b are fixed. The solutions are non more then two. It was shown that the line determined by these two points is perpendicular to the side AB as in the classical case. The above written Diophantian equation is not always solvable. For instance, for its solvability it is necessary the lengths b and c to be of the same parity. Indeed. $c^2 + b^2 - a^2$ must be even, but according to PROPOSITION 2, the same is valid and for $c^2 + b^2 + a^2$, which implies that $c^2 + b^2$ must be even too.

6. A KIND OF QUALITATVE CLASSIFICATION OF DIOPHANTIAN TRIANGLES

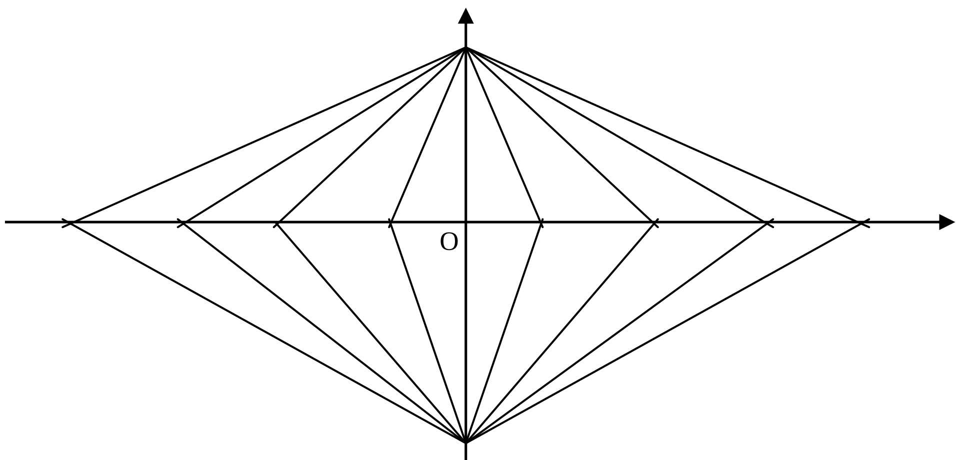
A classification of different types Diophantian, but non-Pythagorean, triangles can be given with the help of the notion of Pythagorean rectangle. This is a rectangle with an inscribed triangle for which the supplementary area is covered by Pythagorean triangles. The sketches below illustrate our idea.



We know different concrete examples of the above exposed sketches. Their uniqueness is conjectured in [2].

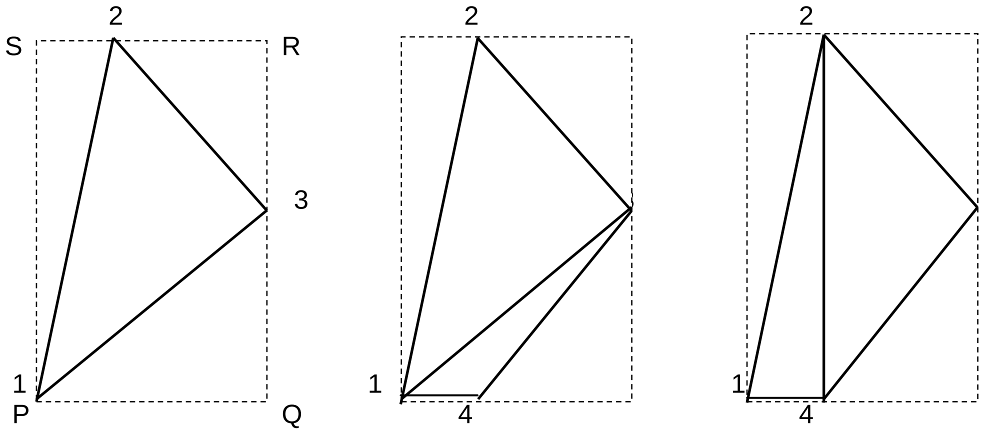
6. SYMMETRIES, DIOPHANTIAN CARPETS

In general Diophantian figures don't admit symmetries. However some of them are symmetric with respect to axes. Example is given below



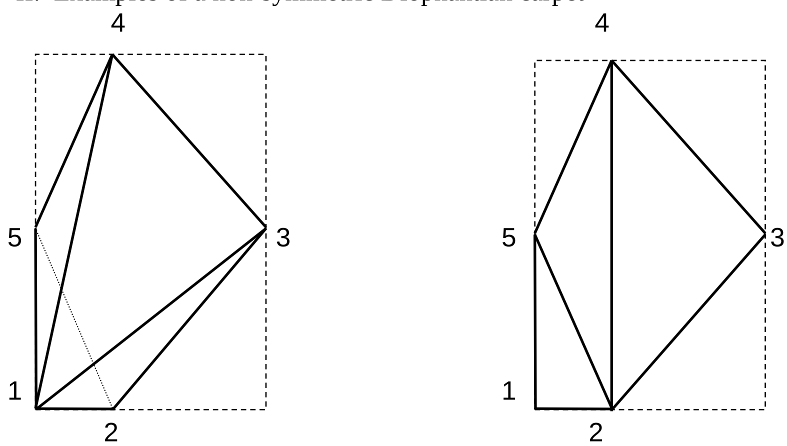
In the next we collect different Diophantian figures and Diophantian carpets. By definition. *Diophantian carpet* is a figure which is equipped with a triangulation by Diophantian triangles, but it is not itself a Diophantian figure.

I. Examples of non-symmetric Diophantian figures



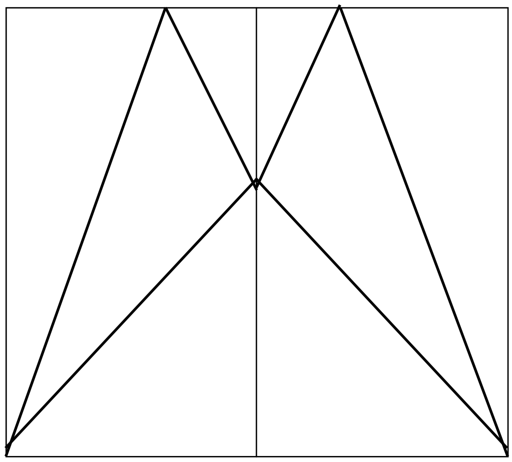
In the first of the above figures the triangles PQ3, 3R2, 2S1 are by definitions Pythagorean, and 123 is a Diophantian one. In the second, we have that π(Q3)=2 (a conjecture!). The figure defined by the points 1, 2, 3 and 4 is a Diophantian figure, equipped with a Diophantian triangulation (the length of the segment 14 is equal to the length of the segment S2). In the third, we have the same Diophantian figure equipped with another Diophantian triangulation.

II. Examples of a non-symmetric Diophantian carpet



Here is exposed the figure 12345 equipped with two different triangulations. In the left the triangulation is composed by Diophantian triangles, which means that the considered figure is a Diophantian carpet, but not a Diophantian figure (the segment 52 is not of integer length).. In the right the triangulation is not Diophantian, as the triangle 125 is not a Diophantian. So the right figure is not a Diophantian carpet.

III. Examples of a symmetric Diophantian carpet



IV. Infinite Diophantian carpets

A simplest infinite Diophantian carpet is defined by a Pythagorean triple (a, b, c), where a and b are considered as legs with the condition $\pi(a) = \pi(b) = 1$.

ACKNOWLEDGENTS

I would like to express my gratitude to Docent Georgi Zlatanov who encourage me to present this paper. I am obliged to Professors Ivan Dimovski and Stancho Dimiev for their advices during my work on the subject.

REFERENCES:

[1] Stancho Dimiev, Ivan Tonov, Diophatian Figures (in Bulgarian), Math. And Math. Education, Publ. Hause of BAS, Sofia, 1986, 383-590, (MR 1988d: 51015).

[2] Maria Brancheva, Stancho Dimiev, Nedelcho Milev, On Diophantian Figures, C.R.Bulg.Acad.Sci. (Submitted for publication)

[3] Trygve Nagell, Introduction to Number Theory, 1950, John Wiley & Sons, Inc., New York

Maria Brancheva, Plovdiv University "Paisy Hilendarsky",
Filial-Smolyan, Dicho Petrov str. No 32, 4700 Smolyan.