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QUASILINEARIZATION OF A BOUNDARY VALUE PROBLEM FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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In this paper we study the existence and approximation of solution for nonlinear impulsive differential equations with fixed moments of impulsive perturbations. Using quasilinearization technique we obtain a monotone sequence of approximate solutions that converges quadratically to a solution.

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Consider the impulsive differential equation

(1) $x' = f(t,x), t \in [0,T], t \neq t_k, k = 1,\dots,p$

(2) $\Delta x\big|_{t=t_k} = I_k(x(t_k)), k = 1,\dots.p$

with boundary condition

(3) $Mx(0) + Nx(T) = C$

where $x\in\mathbf{R}$, f:[0,T]×**R**→**R**, I_k:**R**→**R**, k=1,…,p, M, N, and C are real constants, and $\{t_k\}_{k=1}^{p}$ are fixed points, such that $0<t_1<t_2<\dots<t_p<T$ and $\Delta x\big|_{t=t_k} = x(t_k^+) - x(t_k^-)$.

Let PC([0,T],**R**) denote the set of all functions u:[0,T]→**R** that are piecewise continuous, with points of discontinuity of the first kind t_k, k=1,…,p, at which they are continuous from the left.

Let PC^1([0,T],**R**) denote the set of all functions $u\in$PC([0,T],**R**), that are continuously differentiable at all $t\in[0,T]$, $t\neq t_k$,k=1,…,p, and for which the limits $u'(t_k - 0) = \lim_{t\to t_k^-} u'(t), k = 1,\dots,p$ exist.

Definition 1: *The function* $\alpha\in PC^1([0,T],\mathbf{R})$ ($\beta\in PC^1([0,T],\mathbf{R})$ *is called* ***lower*** (***upper***) ***solution*** *of the boundary value problem* (1) - (3), *if the following inequalities hold*

(4) $\alpha'(t)\le f(t,\alpha(t))$ $(\beta'(t)\ge f(t,\beta(t))$ *for* $t\in[0,T]$, $t\neq t_k$, k=1,…,p,

(5) $\Delta\alpha\big|_{t=t_k} = \alpha(t_k^+) - \alpha(t_k^-) \le I_k(\alpha(t_k))$

$\left(\Delta\beta\big|_{t=t_k} = \beta(t_k^+) - \beta(t_k^-) \ge I_k(\beta(t_k))\right), k = 1,\dots,p$

and the boundary condition

$M\alpha(0)+N\alpha(T)=C$ $(M\beta(0)+N\beta(T)=C)$

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also holds.

Usually, the lower and upper solutions are ordered and either $\alpha(t)\le\beta(t)$ or $\alpha(t)\ge\beta(t)$, $t\in[0,T]$.

Definition 2: *The sequence* $\{u_n\}_{n=0}^{\infty}$ *of functions* $u_n:[0,T]\to\mathbf{R}$ ($u_n\in C[0,T]$) *is called* ***quadratically convergent*** *to the function* $u:[0,T]\to\mathbf{R}$, ($u\in C[0,T]$), *if there exists a constant* $\lambda>0$, *such that*

$$\|u_n - u\| \le \lambda \|u_{n-1} - u\|^2, n = 1,2,...,$$

where $\|u\| = \sup_{t\in[0,T]} |u(t)|$.

Let the functions $\alpha,\beta\in PC^1([0,T],\mathbf{R})$ are such that $\alpha(t)\le\beta(t)$ for $t\in[0,T]$. Define the sets

$S[\alpha,\beta]=\{u\in PC^1([0,T],\mathbf{R}):\alpha(t)\le u(t)\le\beta(t)$ for $t\in[0,T]\}$,

$\Omega[\alpha,\beta]=\{(t,x)\in R^2:t\in[0,T],\ \alpha(t)\le x\le\beta(t)\}$.

We say that the conditions (A) hold, if the following conditions are fulfilled:

A1. The functions $\alpha,\beta\in PC^1([0,T],\mathbf{R})$ are lower and upper solutions respectively of the boundary value problem (1)-(3), such that $\alpha(t)\le\beta(t)$ for $t\in[0,T]$.

A2. The function $f:\Omega[\alpha,\beta]\to\mathbf{R}$ is twice continuously differentiable with respect to its second argument on $\Omega[\alpha,\beta]$.

A3. The functions $I_k:[\alpha(t_k),\beta(t_k)]\to\mathbf{R}$, $k=1,\dots,p$ are twice continuously differentiable in $[\alpha(t_k),\beta(t_k)]$.

A4. The constants M and N are such that

$$MN<0 \text{ and } \left|\frac{N}{M}\right| \le \exp(LT),$$

where $L = \sup\left\{\left| f'_x(t,x)\right| : (t,x) \in \Omega[\alpha,\beta]\right\}$.

A5. The following inequality holds:

$$\int_0^T f'_x(t,x(t))dt \le \delta < 0 \quad \text{for } x\in S[\alpha,\beta].$$

A6. $f''_{xx}(t,x) \ge -2K$, $K>0$ for $(t,x)\in\Omega[\alpha,\beta]$ (6)

A7. $I''_k(x) \ge -2T_k$, $T_k>0$ for $x\in[\alpha(t_k),\beta(t_k)]$, $k=1,\dots,p$. (7)

Define the set

$W=\{(t,x,y)\in\mathbf{R}^3:t\in[0,T],\ \alpha(t)\le y\le x\le\beta(t)\}$

and consider the function $g:W\to\mathbf{R}$ defined by

$$g(t,x,y) = f(t,y) + f'_x(t,y)(x-y) - K(x-y)^2 \quad \text{for } (t,x,y)\in W.$$

Clearly $f(t,x)=g(t,x,x)$ for $(t,x)\in\Omega[\alpha,\beta]$.

Let $t\in[0,T]$ and $x,y,z\in\mathbf{R}$ are such that $\alpha(t)\le z\le x\le y\le\beta(t)$. Then

(8) $g(t,y,z)-g(t,x,z)= f'_x(t,z)(y-x)-K(y+x-2z)(y-x)\ge-(L+2KP)(y-x)$

where $P=\sup\{\beta(t)-\alpha(t): t\in[0,T]\}$.

Define the sets

$W_k=\{(t,x,y)\in\mathbf{R}^2:\alpha(t_k)\le y\le x\le\beta(t_k)\}$, $k=1,\dots,p$

and consider the functions h_k:$W_k \to \mathbf{R}$, $k=1,\dots,p$ defined by

$h_k(x,y)=I_k(y)+I'_k(y)(x-y)-T_k(x-y)^2$, $(x,y)\in W_k$.

Clearly $I_k(x)=h_k(x,x)$ for $x\in[\alpha(t_k),\beta(t_k)]$. Furthermore, if $\alpha(t_k)\le z\le x\le y\le\beta(t_k)$ then

(9) $h_k(y,z)-h_k(x,z)=I'_k(z)(y-x)-T_k(y+x-2z)(y-x)\ge-(S_k+2T_kP)(y-x)$

where $S_k=sup\{|I_k(x)|:x\in[\alpha(t_k),\beta(t_k)]\}$, $k=1,\dots,p$.

We will use the following lemma:

Lemma 1: (Theorem 16.2, [1].) *Assume that for $t\ge t_0$ the following inequality holds*:

$$u(t)\le a(t)+\int_{t_0}^{t} b(s)u(s)ds+\sum_{t_0<t_k<t}\beta_k u(t_k) \quad ,$$

where $u,a,b\in PC(\mathbf{R}_+,\mathbf{R}_+)$, $\beta_k\ge 0$, $k=1,2,\dots$, and $0<t_0<t_1<\dots<t_k<\dots$ are fixed numbers.

Then for $t\ge t_0$ the following inequality holds:

$$u(t)\le a(t)+\int_{t_0}^{t} a(s)b(s)\prod_{s<t_k<t}(1+\beta_k)\exp\left(\int_{t_k}^{t} b(\tau)d\tau\right)ds+$$

$$+\sum_{t_0<t_k<t} a(t_k)\beta_k\prod_{t_k<t_j<t}(1+\beta_j)\exp\left(\int_{t_k}^{t} b(\tau)d\tau\right)$$

Consider the impulsive differential equation

(10) $\dot{X}=g(t,x,\alpha(t))$, $t\in[0,T]$, $t\ne t_k$, $k=1,\dots,p$

(11) $\Delta x\big|_{t=t_k}=h_k(x(t_k),\alpha(t_k))$, $k=1,\dots,p$

We are going to prove the following lemma for existence of a solution of the boundary value problem (10), (11), (3).

Lemma 2: *Suppose that conditions* **A1**-**A4**, **A6**, *and* **A7** *hold. Then the boundary value problem* (10), (11), (3) *has a solution* $x\in S[\alpha,\beta]$.

Proof: From the definition of the function g and inequalities (4) and (6) it follows that for $t\in[0,T]$, $t\ne t_k$, $k=1,\dots,p$ we have the inequalities

(12) $\alpha'(t)\le f(t,\alpha(t))=g(t,\alpha(t),\alpha(t))$

and

$\beta'(t)\ge f(t,\beta(t))=f(t,\alpha(t))+f'_x(t,\beta(t))[\beta(t)-\alpha(t)]+$

$+\frac{1}{2}f''_{xx}(t,\xi(t))[\beta(t)-\alpha(t)]^2\ge g(t,\beta(t),\alpha(t))$,

where $\alpha(t)\le\xi(t)\le\beta(t)$, $t\in[0,T]$.

From the definition of the functions h_k and inequalities (5) and (7) it follows that for $k=1,..,p$ we have the inequalities

(14) $\Delta\alpha\big|_{t=t_k}\le I_k(\alpha(t_k))=h_k(\alpha(t_k),\alpha(t_k))$

and

(15) $\Delta\beta\Big|_{t=t_k} \geq I_k(\beta(t_k))=I_k(\alpha(t_k))+ I_k'(\alpha(t_k))\,[\beta(t_k)-\alpha(t_k)]+$

$+\frac{1}{2}I_k''(\xi_k)\,[\beta(t_k)-\alpha(t_k)]^2\geq h_k(\beta(t_k),\alpha(t_k))$

where $\xi_k\in(\alpha(t_k),\beta(t_k))$.

From (12) -(15) it follows that the functions α and β are lower and upper solutions respectively of the boundary value problem (10), (11), (3).

Define the operator F on S[α,β] by the equality

(16) $Fx(t)=CA\,exp[-(L+2KP)t]+\int_0^T G(t,s)q(t,x(s))ds +$

$+\sum_{k=1}^{p} G(t,t_k)h_k(x(t_k),\alpha(t_k))$

for $x\in S[\alpha,\beta]$ and $t\in[0,T]$, where

(17) $A=[M+N\,exp(-(L+2KP)T)]^{-1}$,

$q(t,u)=g(t,u,\alpha(t))+(L+2KP)u$, $(t,u)\in\Omega[\alpha,\beta]$

(18) $G(t,s)=\begin{cases} AM\exp[-(L+2KP)(t-s)], 0\leq s<t\leq T \\ (AM-1)\exp[-(L+2KP)(t-s)], 0\leq t\leq s\leq T \end{cases}$

Clearly if $x\in S[\alpha,\beta]$ and $x(t)=Fx(t)$ for $t\in[0,T]$ (i.e. x is a fixed point for the operator F), then $x=x(t)$ is a solution of the boundary value problem (10), (11), (3).

We are going to show that $G(t,s)\geq 0$ for $t,s\in[0,T]$.

Indeed, from (17) it follows that

$A=[M(1-\left|\frac{N}{M}\right| exp(-(L+2KP)T))]^{-1}$, or $AM[1-\left|\frac{N}{M}\right| exp(-(L+2KP)T)]=1$

By **A4** we have that

$\left|\frac{N}{M}\right| exp(-(L+2KP)T)\leq 1$

Therefore AM>0 and $AM-1=-AN\exp(-(L+2KP)T)\geq 0$, and from these inequalities, using (18), we have that $G(t,s)\geq 0$ for $t,s\in[0,T]$. From inequality $G(t,s)\geq 0$ and the inequality (8) and (9) it follows that the operator F is monotone non-decreasing, i.e. for $x,y\in S[\alpha,\beta]$ and $x(t)\leq y(t)$ for $t\in[0,T]$we have $Fx(t)\leq Fy(t)$ for $t\in[0,T]$.

Therefore $F:S[\alpha,\beta]\to S[\alpha,\beta]$, and by Shauder's Theorem there exists a function $x\in S[\alpha,\beta]$ such that $x=Fx$. As we noted before this means that $x=x(t)$ is a solution to the boundary value problem (10), (11), (3).

This complites the proof of Lemma 2.�

We are going to justify a method for constructing a monotone sequence of piecewise continuous functions that is quadratically convergent to a solution of the boundary value problem (1) - (3).

Theorem 1: *Suppose conditions* (A) *hold*.

Then there exists a sequence $\{u_n\}_{n=0}^{\infty}$ *of functions* $u_n \in PC^1([0,T],\mathbf{R})$, $n=1,2,\ldots$, *such that* $u_0(t) \le u_1(t) \le \ldots \le u_n(t) \le \ldots$, *which is quadratically convergent to a solution u of the boundary problem* (1) - (3).

Proof: Set $u_0=\alpha$. By Lemma 2 we have that there exists a solution $u_1 \in S[\alpha,\beta]$ of the boundary value problem (10), (11), (3).

Suppose that we have defined the functions $u_0,u_1,\ldots,u_{n-1} \in PC^1([0,T],\mathbf{R})$ such that $u_j \in S[u_{j-1},\beta]$, $j=1,\ldots,n-1$ and

$u_0(t) \le u_1(t) \le \ldots \le u_{n-1}(t)$, $t \in [0,T]$

where the function u_j, $j=1,\ldots,n-1$ is a solution of the boundary value problem

(19) $x' = g(t,x,u_{j-1}(t))$, $t \in [0,T]$, $t \ne t_k$, $k=1,\ldots,p$,

(20) $\Delta x\big|_{t=t_k} = h_k(x(t_k),u_{j-1}(t_k))$, $k=1,\ldots,p$

(21) $Mx(0)+Nx(T)=c$

From the definitions of the functions g, h and u_{n-1} and inequality (6) it follows that for $t \in [0,T]$, $t \ne t_k$, $k=1,\ldots,p$ we have the following inequality

$g(t,u_{n-1}(t),u_{n-1}(t))=f(t,u_{n-1}(t)) \ge f(\mathrm{i},u_{n-2}(t))+ f'_x(t,u_{n-2}(t))[u_{n-1}(t)-u_{n-2}(t)]-$

$-K[u_{n-1}(t)-u_{n-2}(t)]^2=g(t,u_{n-1}(t),u_{n-2}(t))= u'_{n-1}(t)$

i.e.

(22) $u'_{n-1}(t) \le g(t,u_{n-1}(t),u_{n-1}(t))$, $t \in [0,T]$, for $t \ne t_k$, $k=1,\ldots,p$.

From the definitions of the functions h_k and u_{n-1} and inequality (7) it follows that for $k=1,\ldots,p$ we have the following inequalities

$h_k(u_{n-1}(t_k),u_{n-1}(t_k))=I_k(u_{n-1}(t_k)) \ge I_k(u_{n-2}(t_k))+ I'_k(u_{n-2}(t_k))[u_{n-1}(t_k)-u_{n-2}(t_k)]-$

$-T_k[u_{n-1}(t_k)-u_{n-2}(t_k)]^2=h_k(u_{n-1}(t_k),u_{n-2}(t_k))= \Delta u_{n-1}\big|_{t=t_k}$,

i.e.

(23) $\Delta u_{n-1}\big|_{t=t_k} \le h_k(u_{n-1}(t_k),u_{n-1}(t_k))$, $k=1,\ldots,p$.

By (22) and (23) we have that the function u_{n-1} is a lower solution of the boundary problem (19) - (21) for $j=n$.

Furthermore from inequalities (4)-(7) we obtain

$\beta'(t) \ge f(t,\beta(t)) \ge g(t,\beta(t),u_{n-1}(t))$, $t \in [0,T]$, $t \ne t_k$, $k=1,\ldots,p$,

$\Delta\beta\big|_{t=t_k} \ge I_k(\beta(t_k)) \ge h_k(\beta(t_k),u_{n-1}(t_k))$, $k=1,\ldots,p$

which shows that the function β is an upper solution of the boundary value problem (17)-(19) for $j=n$.

By Lemma 2 there exists a solution $u_n \in S[u_{n-1},\beta]$ of the boundary value problem (19)-(21) for $j=n$. Furthermore, u_n is a fixed point of the operator F, defined on $S[u_{n-1},\beta]$ by

$$Fx(t)=CA\,exp[-(L+2KP)t]+$$
$$+\int_0^T G(t,s)q_n(s,x(s))ds+\sum_{k=1}^{p} G(t,t_k)h_k(x(t_k),u_{n-1}(t_k))$$

where the number A is determined from(17), the function G is defined by (18), and

$q_n(t,x)=g(t,x,u_{n-1}(t))+(L+2KP)x$ for $(t,x)\in\Omega[u_{n-1},\beta]$.

This means that for $t\in[0,T]$ we have

$$(24)\quad u_n(t)=Fu_n(t)=CA\,exp[-(L+2KP)t]+\int_0^T G(t,s)q_n(t,u_n(s))ds+$$
$$+\sum_{k=0}^{p} G(t,t_k)h_k(u_n(t_k),u_{n-1}(t_k))$$

Thus we have constructed a sequence $\{u_n\}_{n=0}^{\infty}$ of functions $u_n\in PC^1([0,T],\mathbf{R})$ such that $u_n\in S[u_{n-1},\beta]$, $n=1,2,\ldots$, for which

$u_0(t)\le u_1(t)\le\ldots\le u_n(t)\le\ldots$, $t\in[0,T]$.

Clearly there exists a function $u\in PC^1([0,T],\mathbf{R})$ such that

$\lim_{n\to\infty} u_n(t)=u(t)$, $t\in[0,T]$

From (24) it follows that the sequence $\{u_n\}_{n=0}^{\infty}$ is uniformly convergent to the function u on every finite closed interval, which does not contain t_k, $k=1,\ldots,p$, and that

$\lim_{n\to\infty} q_n(t,u_n(t)) =g(t,u(t),u(t))+(L+2KP)u(t)=f(t,u(t))+(L+2KP)u(t)$, $t\in[0,T]$.

Therefore the function $u=u(t)$ is a solution of the boundary value problem

$x'+(L+2KP)x=f(t,u(t))+(L+2KP)u(t)$, $t\in[0,T]$, $t\ne t_k$, $k=1,\ldots,p$,

$\Delta x\big|_{t=t_k}=I_k(x(t_k))$, $k=1,\ldots,p$

$Mx(0)+Nx(t)=C$

and that means that $u=u(t)$ is a solution of the boundary value problem (1)-(3).

Finally we show that the sequence $\{u_n\}_{n=1}^{\infty}$ is quadratically convergent to the function u.

Consider the function $F:\Omega[\alpha,\beta]\to\mathbf{R}$, defined by

$F(t,x)=f(t,x)+Kx^2$, $(t,x)\in\Omega[\alpha,\beta]$.

From condition **A2** and inequality (6) it follows that there exists a constant Q>0 such that

(25) $0\le F''_{xx}(t,x)\le Q$ for $(t,x)\in\Omega[\alpha,\beta]$

Let $z_n(t)=u(t)-u_n(t)$, $t\in[0,T]$, $n=1,2,\ldots$. Using the fact that $u=u(t)$ is a solution of the boundary value problem (1)-(3), and $u_n=u_n(t)$ is a solution of the boundary value problem (19)-(21) for $j=n$, we obtain

$$(26)\quad z'_n(t)=f(t,u(t))-g(t,u_n(t),u_{n-1}(t))=$$
$$=f(t,u(t))-f(t,u_{n-1}(t))-F'_x(t,u_{n-1}(t))[u_n(t)-u_{n-1}(t)]+$$
$$+K[u_n^2(t)-u_{n-1}^2(t)],\ t\in[0,T],\ t\ne t_k,\ k=1,\ldots,p$$

Using (25) and (26), the Mean Value Theorem and the definition of the function F, we obtain that there exist functions ξ and η, $u_{n-1}(t)\le\xi(t)\le u(t)$, $u_{n-1}(t)\le\eta(t)\le\xi(t)$, $t\in[0,T]$ such that

(27) $z_n'(t)= f_x'(t,\xi(t))[u(t)-u_{n-1}(t)]- F_x'(t,u_{n-1}(t))[u_n(t)-u_{n-1}(t)]+$

$+K[u_n^2(t)-u_{n-1}^2(t)]\le[F_x'(t,\xi(t))- F_x'(t,u_{n-1}(t))+K(u(t)-u_{n-1}(t))](u(t)-$

$-u_{n-1}(t))+[F_x'(t,u_{n-1}(t))-K(u(t)+u_n(t))](u(t)-u_n(t))=$

$=F_{xx}''(t,\eta(t))(\xi(t)-u_{n-1}(t))(u(t)-u_{n-1}(t))+\psi_n(t)(u(t)-u_n(t))+K(u(t)-u_{n-1}(t))^2\le$

$\le(Q+K)z_{n-1}^2(t)+\psi_n(t)z_n(t),\ t\in[0,T],\ t\ne t_k,\ k=1,\dots,p$

where $\psi_n(t)=F_x'(t,u_{n-1}(t))-K(u(t)+u_n(t))$.

Consider the functions $G_k:[\alpha(t_k),\beta(t_k)]\to\mathbf{R}$, $k=1,\dots,p$, defined by
$G_k(x)=I_k(x)+T_kx^2,\ x\in[\alpha(t_k),\beta(t_k)]$.

From condition **A3** and inequalities (7) it follows that there exist constants $Q_k>0$ such that

(28) $0\le G_k''(x)\le Q_k$ for $x\in[\alpha(t_k),\beta(t_k)]$.

Again we use that $u=u(t)$ is a solution of the boundary value problem (1)-(3) and $u_n=u_n(t)$ is a solution of the boundary value problem (19)-(21) for $j=n$ and we obtain

(29) $\Delta z_n\big|_{t=t_k}=I_k(u(t_k))-h_k(u_n(t_k),u_{n-1}(t_k))=$

$=I_k(u(t_k))-I_k(u_{n-1}(t_k))- G_k'(u_{n-1}(t_k))[u_n(t_k)-u_{n-1}(t_k)]+$

$+T_k[u_n^2(t_k)-u_{n-1}^2(t_k)]$

Using (28) and (29), the Mean Value Theorem and the definitions of the functions G_k, we obtain that there exists constants $\xi_k\in[u_{n-1}(t_k),u(t_k)]$ and $\eta_k\in[u_{n-1}(t_k),\xi_k]$, such that

(30) $\Delta z_n\big|_{t=t_k}= I_k'(\xi_k)[u(t_k)-u_{n-1}(t_k)]- G_k'(u_{n-1}(t_k))[u_n(t_k)-u_{n-1}(t_k)]+$

$+T_k[u_n^2(t_k)-u_{n-1}^2(t_k)]\le$

$\le[G_k'(\xi_k)- G_k'(u_{n-1}(t_k))+T_k(u(t_k)-u_{n-1}(t_k))](u(t_k)-u_{n-1}(t_k))+$

$+[G_k'(u_{n-1}(t_k))-T_k(u(t_k)-u_n(t_k))](u(t_k)-u_n(t_k))=$

$=G_k''(\eta_k)(\xi_k-u_{n-1}(t_k))(u(t_k)-u_{n-1}(t_k))+b_k^{(n)}(u(t_k)-u_n(t_k))+T_k(u(t_k)-u_{n-1}(t_k))^2\le$

$\le(Q_k+T_k)z_n^2(t_k)+b_k^{(n)}z_n(t_k),$

where $b_k^{(n)}=G_k'(u_{n-1}(t_k))-T_k(u(t_k)-u_n(t_k))$.

By (27) and (30) the functions z_n satisfy the impulsive differential inequalities

(31) $z_n'(t)\le\psi_n(t)z_n(t)+(Q+K)z_n^2(t),\ t\in[0,T],\ t\ne t_k,\ k=1,\dots,p$

(32) $\Delta z_n\big|_{t=t_k}\le b_k^{(n)}z_n(t_k)+(Q_k+T_k)z_n^2(t_k),\ k=1,\dots,p$

(33) $Mz_n(0)+Nz_n(T)=C$

By Lemma 1 we have

(34) $z_n(t)\le z_n(0)\prod_{0\le t_k<t}(1+b_k^{(n)})\,\sigma(t)+\int_0^t\prod_{s<t_k<t}(1+b_k^{(n)})\,\sigma(t)\sigma^{-1}(s)(Q+K)z_n^2(s)ds+$

$+\sum_{0<t_k<t}\prod_{t_k<t_j<t}(1+b_j^{(n)})\sigma(t)\sigma^{-1}(t_k)(Q_k+T_k)z_n^2(t_k)$, $t\in[0,T]$

where $\sigma(t)=exp\left(\int_0^t \psi_n(s)ds\right)$.

By the boundary condition (33) and inequality (34) it follows that for $t\in[0,T]$ the following inequality holds:

(35) $z_0=z_n(0)$

$$\leq\left[\frac{M}{N}+\prod_{k=1}^{p}(1+b_k^{(n)})\sigma(T)\right]^{-1}\left\{\int_0^T \prod_{s<t<T}(1+b_j^{(n)})\sigma(T)\sigma^{-1}(s)(Q+K)z_n^2(s)ds+\right.$$

$$\left.+\sum_{k=1}^{p}\prod_{t_k<t_j<T}(1+b_j^{(n)})\sigma(T)\sigma^{-1}(t_k)(Q_k+T_k)z_n^2(t_k)\right\}$$

From inequalities (34) and (35) it follows that there exists a constant $\lambda>0$ such that

$sup\{|z_n(t)|:t\in[0,T]\}\leq\lambda sup\{|z_{n-1}(t)|:t\in[0,T]\}^2$,

which means that

$||u-u_n||\leq\lambda||u-u_{n-1}||^2$.

This concludes the proof of Theorem 1. �

Remark: In the case when the boundary value problem (1)-(3) is without impulses, i.e. $I_k(x)\equiv 0$, $k=1,\dots,p$ and M=-N=1, C=0 (i.e. we have a periodic problem for ordinary differential equations), the results in this paper are identical with the results in [2].

Consider the impulsive differential equation (1), (2) with periodic boundary condition

(36) $x(0)=x(T)$

Then we have the following corollary of Theorem 1:

Theorem 2: *Suppose conditions* **A1**-**A3**, *and* **A5** *hold. Then there exists a monotone increasing sequence* $\{u_n\}_{n=0}^{\infty}$ *of functions* $u_n\in PC^1([0,T],\mathbf{R})$, *which is quadratically convergent to a solution u of the periodic problem* (1), (2), (36).

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