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NUMERICAL RESULTS OF SOME MAGNITUDES OVER SETS OF CONJUGATE FUNCTIONS

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The paper is devoted to the calculus of numerical values and the graphical mappings of some magnitudes over sets of conjugate functions. They can be applied for the solution of an extremal problem.[1]

Introduction.

Let L (L_∞) be the space of the defined on the real axis 2π-periodic summable (essentially limited) real-valued functions with norm

$$(0.1)\qquad \|f\|_L = \|f\|_1 = \int_0^{2\pi} |f(x)|\,dx \quad ; \quad \|f\|_{L_\infty} = \|f\|_\infty = \sup_x vrai\, |f(x)| \ .$$

Let $f \sim \dfrac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$; $\tilde{f} \sim \sum_{k=1}^{\infty}(-b_k \cos kx + a_k \sin kx)$ be the Fourier series expansions to $f(x)$ and to the trigonometrically conjugate to $f(x)$ function $\tilde{f}(x)$ respectively [1]. During the last decade our efforts have been intent on the solution of the following extremal problem [2]:

$$(0.2)\qquad \sup_{\|f\|_\infty \le 1} \|\tilde{f}\|_1 = 4\tilde{K}_1 \ ,$$

where

$$(0.3)\qquad \tilde{K}_1 = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(2\nu+1)^2} = \int_0^\infty \arctan\frac{1}{\sinh(\pi y/2)}\,dy = 1.166243616...$$

is Favard's constant. This fact is closely linked with a generalization of the theorem of Stein and Weiss concerning metric properties of the conjugate characteristic functions of given sets on the interval $[0,2\pi]$. This extension was made in our work [2] (th.1.2.).

The current paper appears to be an application of our theoretical results in [2], [3].

Statement.

Further on we shall use the symbols and results of the articles [2], [3]. The pivotal role for the solution of the problem (0.2),(0.3) fell to the following magnitudes:

$$(1.1)\qquad F\left[|\tilde{g}|;E\right] = \int_0^\infty \mu\left[|\tilde{g}| > y\right] \cap E\,dy \quad , \quad (g \in \Omega)$$

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[1] AMS Subject Classification: 42A50, 28A25

(1.2) $$G(\tilde{g};a) = 2\pi^{-1}\int_0^\infty dy \int_E arctg\{[\cosh(\pi|\tilde{g}(t)|/2)]/[\sinh(\pi y/2)]\}\, dt\,,$$

where $\Omega = \Omega(E)$ and $a = \mu E$ are determined in [2,3].

For the principal functions $g_k(x)$ introduced in items k^0 (k=1,2) in [3], (see also examples 1;2 in [2]) we have proved

(1.3) $$F[\|\tilde{g}_1|;E] = 4\pi^{-1}A(a/2) \equiv \Psi_1(a)\,,$$

(1.4) $$G(\tilde{g}_1;a) = 4\tilde{K}_1 + 4\pi^{-1}\{A(a/2) - 2B(\gamma) - A(\gamma + a/2) - A(\gamma - a/2)\}\,,$$

(1.5) $$F[|\tilde{g}_2|;E] = 4\pi^{-1}\{A(a/2) + B(a/2)\} \equiv \Psi_2(a)$$

(1.6) $$G(\tilde{g}_2;a) = 4\tilde{K}_1 + 4\pi^{-1}\{A(a/2) + B(a/2) - 2[A(\pi/2 - a/4) + B(\pi/2 - a/4)]\}$$

where $\gamma = \gamma(a) = \arccos[-\sin^2(a/4)]$ $(0 < a < 2\pi)$ and $A(x), B(x)$ are the Fourier expansions (1.3) in [3].

Here we shall provide some numerical data for the magnitudes $F[\|\tilde{g}_k|;E]$ and $G(\tilde{g}_k;a)$ (k=1,2) and their graphs. (Throughout the whole note the values of the magnitudes in suitable points are carried out with an accuracy of eight decimal digits).

Table I.
Numerical data of the magnitudes $F[\|\tilde{g}_k|;E]$ and $G(\tilde{g}_k;a)$ (k=1,2)

Point / Magn.	$a = \pi/12$	$a = \pi/6$	$a = 2\pi/3$	$a = \pi$
$\Psi_1(a)$	0.505593657	0.780376339	1.292263788	$\tilde{K}_1$
$G(\tilde{g}_1;a)$	0.516494189	0.823885669	1.961030926	2.600558609
$\Psi_2(a)$	0.620999145	1.010471833	2.153772982	$2\tilde{K}_1$
$G(\tilde{g}_2;a)$	0.626455248	1.032319690	2.511201484	3.163071510

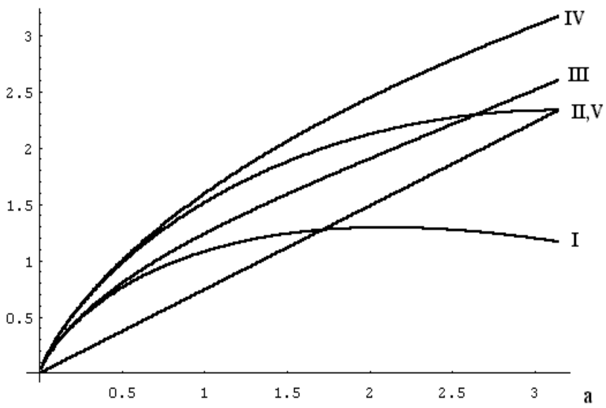

Fig.1. Graphs of the functions: I) $\Psi_1(a)$; II) $\Psi_2(a)$; III) $G(\tilde{g}_1;a)$; IV) $G(\tilde{g}_2;a)$; V) $2\pi^{-1}\tilde{K}_1 a$.

So all properties of the magnitudes $\Psi_k(a)$ and $G(\tilde{g}_k; a)$ (k=1,2) are getting apparently true. They are stated in [3] (see also [2] for more general case). Here we shall emphasize only on the following inequalities

(1.7) $\quad 0 < G(\tilde{g}_2; a) - \Psi_2(a) < G(\tilde{g}_1; a) - \Psi_1(a) \quad (0 < a < 2\pi)$

and their graphical mapping.

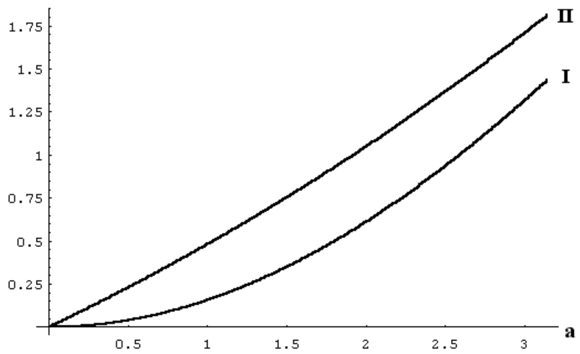

Fig.2. Graphs of the functions: I) $G(\tilde{g}_2; a) - \Psi_2(a)$; II) $G(\tilde{g}_1; a) - \Psi_1(a)$.

Finally, as an application, let illustrate our theoretical results by diverse numerical examples. For brevity we make use of the following notations: $a = \mu E \in (0, 2\pi)$; $\tilde{K}_1 / 3 = 0.388747872$; $2\tilde{K}_1 = 2.332487232$; $4\tilde{K}_1 / 3 = 1.554991488$. In this examples we indicate only the functions. Data for the values of the magnitudes mentioned above are shown in Table II.

$g_1(t) = \{+1 \text{ for } t \in (0, 11\pi/12);\ -1 \text{ for } t \in (11\pi/12, 11\pi/6);$
$\quad 0 \text{ for } t \in (11\pi/6, 2\pi)\}, \qquad (a = \pi/6)$

$g_2(t) = \{+1 \text{ for } t \in (0, 11\pi/12);\ -1 \text{ for } t \in (\pi, 23\pi/12);$
$\quad 0 \text{ for } t \in (11\pi/12, \pi) \cup (23\pi/12, 2\pi)\}, \qquad (a = \pi/6)$

$g_3(t) = \{+1 \text{ for } t \in (0, \pi/12);\ -1 \text{ for } t \in (\pi/12, 11\pi/6);\ 0 \text{ for } t \in (11\pi/6, 2\pi), \quad (a = \pi/6)$

$g_4(t) = \{+1 \text{ for } t \in (0, \pi/6);\ -1 \text{ for } t \in (\pi/6, 11\pi/6);\ 0 \text{ for } t \in (11\pi/6, 2\pi), \ (a = \pi/6)$

$g_5(t) = \{+1 \text{ for } t \in (0, 2\pi/3);\ -1 \text{ for } t \in (\pi, 5\pi/3);$
$\quad 0 \text{ for } t \in (2\pi/3, \pi) \cup (5\pi/3, 2\pi)\}, \qquad (a = 2\pi/3)$

$g_6(t) = \{+1 \text{ for } t \in (0, \pi/3) \cup (\pi, 4\pi/3);\ -1 \text{ for } t \in (\pi/2, 5\pi/6) \cup$
$\quad (3\pi/2, 11\pi/6);\ \ 0 \text{ for } t \in (\pi/3, \pi/2) \cup (5\pi/6, \pi) \cup$
$\quad (4\pi/3, 3\pi/2) \cup (11\pi/6, 2\pi)\}, \quad (a = 2\pi/3)$

$g_7(t) = \{+1 \text{ for } t \in (0, 2\pi/9) \cup (2\pi/3, 8\pi/9) \cup (4\pi/3, 14\pi/9);$
$\quad -1 \text{ for } t \in (\pi/3, 5\pi/9) \cup (\pi, 11\pi/9) \cup (5\pi/3, 17\pi/9);$
$\quad 0 \text{ for } t \in (2\pi/9, \pi/3) \cup (5\pi/9, 2\pi/3) \cup (8\pi/9, \pi) \cup$
$\quad (11\pi/9, 4\pi/3) \cup (14\pi/9, 5\pi/3) \cup (17\pi/9, 2\pi)\}, \quad (a = 2\pi/3)$

$g_8(t)=\{+1$ for $t\in(0,\pi/6)\bigcup(2\pi/3,7\pi/6)$; -1 for $t\in(\pi/6,\pi/3)\bigcup$
$(3\pi/2,2\pi)$; 0 for $t\in(\pi/3,2\pi/3)\bigcup(7\pi/6,3\pi/2)\}$, $(a=2\pi/3)$

$g_9(t)=\{+1$ for $t\in(0,4\pi/9)$; -1 for $t\in(7\pi/9,11\pi/9)\bigcup(14\pi/9,2\pi)$;
0 for $t\in(4\pi/9,7\pi/9)\bigcup(11\pi/9,14\pi/9)$, $(a=2\pi/3)$

$g_{10}(t)=\{+1$ for $t\in(0,\pi/3)$; -1 for $t\in(2\pi/3,7\pi/6)\bigcup(3\pi/2,2\pi)$;
0 for $t\in(\pi/3,2\pi/3)\bigcup(7\pi/6,3\pi/2)\}$, $(a=2\pi/3)$

$g_{11}(t)=\{+1$ for $t\in(0,2\pi/3)$; -1 for $t\in(5\pi/6,3\pi/2)$;
0 for $t\in(2\pi/3,5\pi/6)\bigcup(3\pi/2,2\pi)\}$, $(a=2\pi/3)$

$g_{12}(t)=\{+1$ for $t\in(0,\pi/3)\bigcup(10\pi/9,13\pi/9)$;
-1 for $t\in(5\pi/9,8\pi/9)\bigcup(15\pi/9,2\pi)$;
0 for $t\in(\pi/3,5\pi/9)\bigcup(8\pi/9,10\pi/9)\bigcup(13\pi/9,15\pi/9)\}$, $(a=2\pi/3)$

$g_{13}(t)=\{+1$ for $t\in(0,4\pi/9)\bigcup(4\pi/3,16\pi/9)$; -1 for $t\in(2\pi/3,10\pi/9)$;
0 for $t\in(4\pi/9,2\pi/3)\bigcup(10\pi/9,4\pi/3)\bigcup(16\pi/9,2\pi)\}$, $(a=2\pi/3)$

$g_{14}(t)=\{+1$ for $t\in(0,\pi/2)$; -1 for $t\in(\pi,3\pi/2)$;
0 for $t\in(\pi/2,\pi)\bigcup(3\pi/2,2\pi)\}$, $(a=\pi)$

$g_{15}(t)=\{+1$ for $t\in(0,\pi/4)\bigcup(\pi,5\pi/4)$;
-1 for $t\in(\pi/2,3\pi/4)\bigcup(3\pi/2,7\pi/4)$; 0 for $t\in$
$(\pi/4,\pi/2)\bigcup(3\pi/4,\pi)\bigcup(5\pi/4,3\pi/2)\bigcup(7\pi/4,2\pi)\}$, $(a=\pi)$

$g_{16}(t)=\{+1$ for $t\in(0,\pi/6)\bigcup(2\pi/3,5\pi/6)\bigcup(4\pi/3,3\pi/2)$;
-1 for $t\in(\pi/3,\pi/2)\bigcup(\pi,7\pi/6)\bigcup(5\pi/3,11\pi/6)$;
0 for $t\in(\pi/6,\pi/3)\bigcup(\pi/2,2\pi/3)\bigcup(5\pi/6,\pi)\bigcup$
$(7\pi/6,4\pi/3)\bigcup(3\pi/2,5\pi/3)\bigcup(11\pi/6,2\pi)\}$, $(a=\pi)$

Table II. Numerical data of the magnitudes.

Ex.	a	$2\pi^{-1}a\tilde{K}_1$	$F[\lvert\tilde{g}\rvert;E]$	$G(\tilde{g};a)$	$R(\tilde{g},E)$	$\lVert\tilde{g}\rVert_1$
$g_1(t)$	$\pi/6$	$\tilde{K}_1/3$	0.780376339	0.823885667	0.043509328	4.621465136
$g_2(t)$	$\pi/6$	$\tilde{K}_1/3$	1.010471833	1.032319690	0.021847857	4.643126607
$g_3(t)$	$\pi/6$	$\tilde{K}_1/3$	0.315389790	0.491006481	0.175616691	1.845326974
$g_4(t)$	$\pi/6$	$\tilde{K}_1/3$	0.454429797	0.570119709	0.115689912	2.507861939
$g_5(t)$	$2\pi/3$	$4\tilde{K}_1/3$	2.153772982	2.511201482	0.357428500	4.307545964
$g_6(t)$	$2\pi/3$	$4\tilde{K}_1/3$	2.153772979	2.511201480	0.357428501	4.307545963
$g_7(t)$	$2\pi/3$	$4\tilde{K}_1/3$	2.153772985	2.984357253	0.830584268	3.834390196
$g_8(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.779534547	2.253078231	0.473543684	4.191430780

$g_9(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.258459280	2.007423477	0.748964199	3.759261739
$g_{10}(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.067788863	1.639078572	0.571289709	3.736256251
$g_{11}(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.994911846	2.428460953	0.433549107	4.231425357
$g_{12}(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.838654734	2.289879847	0.451225113	4.213749351
$g_{13}(t)$	$2\pi/3$	$4\tilde{K}_1/3$	1.844577273	2.380464042	0.535886769	3.972339167
$g_{14}(t)$	π	$2\tilde{K}_1$	$2\tilde{K}_1$	3.163071520	0.830584278	3.834390176
$g_{15}(t)$	π	$2\tilde{K}_1$	$2\tilde{K}_1$	3.163071520	0.830584275	3.834390189
$g_{16}(t)$	π	$2\tilde{K}_1$	$2\tilde{K}_1$	2.689915730	0.357428498	4.307545966

Here are the graphs of some interesting conjugate functions, shown in Fig.3-Fig.5.

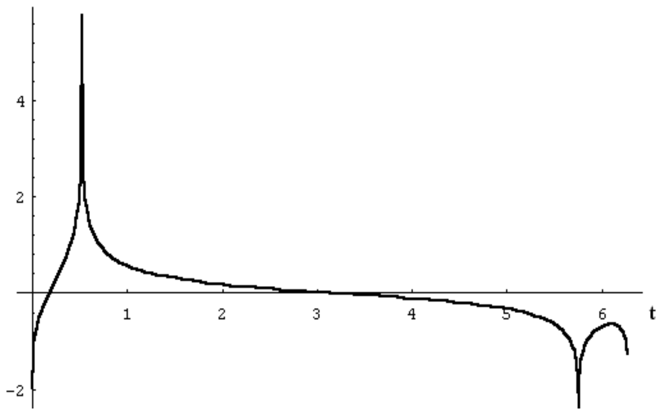

Fig.3. Graph of the conjugate function $\tilde{g}_4(t)$.

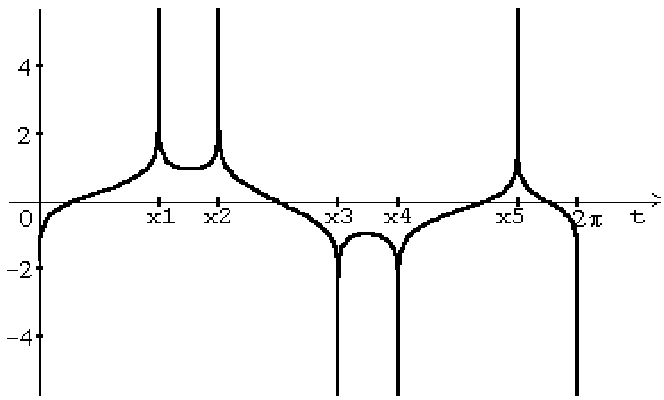

Fig.4. Graph of the conjugate function $\tilde{g}_{13}(t)$.

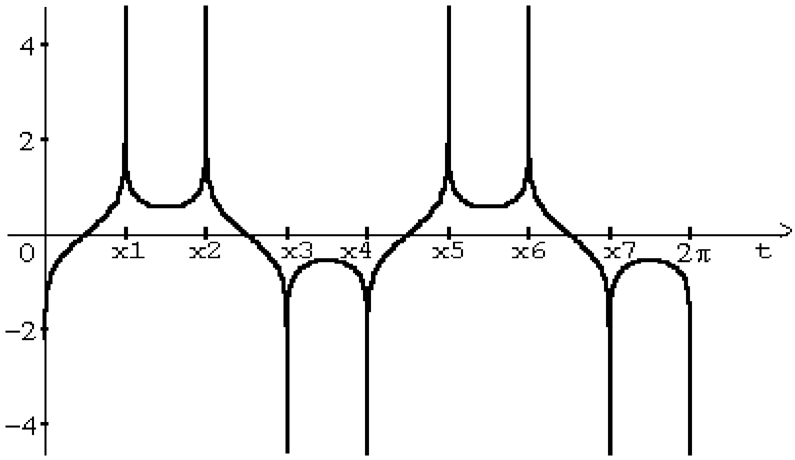

Fig.5. Graph of the conjugate function $\tilde{g}_{15}(t)$.

Conclusion.

The numerical results presented in this paper are only the part of our vast investigation on the extremal problem (0.2), (0.3). We hope the graphs of some interesting conjugate functions, shown in Fig.3-Fig.5, will certainly contribute to more clearness in discussions over the indicated task.

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