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EXPONENTIAL POLYSTABILITY OF A FLAG OF INVARIANT SETS

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We analyse exponential polystability of a flag of invariant sets using Lyapunov vector-functions and comparison system.

1. Introduction

Corduneanu C.[3] proves exponential stability theorem on a part of the variables using Lyapunov function and comparison system. In his work [6] Martynyuk A. A. gives new method for the solution of exponential stability problem on a part ot the variables and in [2] and also in [5] considers exponential polystability on a part of the variables.

In this paper the authors use the idea for polystability of a flag of invariant sets propounded by Russinov I.K.[4] and analyses it through Lyapunov vector-function [7].

2. Preliminary results

We consider the differential system

(1) $\dot{x} = X(t,x), \ x(t_0) = x_0$,

where $x \in R^n$, $X(t,x) \in C(R^+ x R^n, R^n)$ and $X(t,0) \equiv 0$ for each $t \in R^+$ $(R^+ = [0,+\infty))$

We decompose the vector $x \in R^n$ into three subvectors $x^{(i)} \in R^{n_i}$, i=1,2,3, $n = \sum_{i=1}^{3} n_i$,

$R^n = R^{n_1} \oplus R^{n_2} \oplus R^{n_3}$

We write the system (1) in the form

(2) $\dot{x}^{(i)} = X_i(t, x^{(1)}, x^{(2)}, x^{(3)}), \ x^{(i)}(t_0) = x_0^{(i)}$, i = 1,2,3.

We make the following notations and norms:

$$\left\| x^{(i)} \right\| = \left(\sum_{j=1}^{n_i} x_j^2 \right)^{1/2}, \ \left\| x^{(k,s)} \right\| = \left(\left\| x^{(k)} \right\|^2 + \ldots + \left\| x^{(s)} \right\|^2 \right)^{1/2},$$

i=1,2,3, k,s $\in \{1,2,3\}$, k<s, $\|x\| = \left(\sum_{\tau=1}^{n} x_\tau^2 \right)^{1/2} = \left(\sum_{i=1}^{3} \left\| x^{(i)} \right\|^2 \right)^{1/2}$

We assume that:

a) the right parts of the system(1) (respectively(2)) satisfy the condition for uniqueness of solution (for example the local condition of Lipschitz on x);

b) the solutions of the system(2) are $(x^{(1)}, x^{(3)})$ - prolongable [1].

Similary we assume that $X_1\left(t,0,0,x^{(3)}\right)\equiv 0$ and $X_2\left(t,x^{(1)},0,x^{(3)}\right)\equiv 0$. Then the system(2) has a flag of invariant sets[4] $\left\{M_{(1,2)},M_{(2)}\right\}$, where $M_{(1,2)}=\left\{x:x^{(1,2)}=0\right\}$ and $M_2=\left\{x:x^{(2)}=0\right\}$.

Definition 1. The flag of invariant sets $\left\{M_{(1,2)},M_{(2)}\right\}$ of the system(2) is exponentially polystable when:

1) The invariant set $M_{(1,2)}$ of the system(2) is exponentially stable, i.e. if there exists a real number $\lambda>0$ and for each $\varepsilon>0$ there exists a function $\delta(\varepsilon)$ such that $\left\|x_0^{(1,2)}\right\|<\delta(\varepsilon)$ (0< $\left\|x_0^{(3)}\right\|<+\infty$) implies $\left\|x^{(1,2)}(t;t_0,x_0)\right\|\leq$ $\leq \varepsilon \exp$ [- $\lambda(t-t_0)$] for each $t\geq t_0$ [6];

2) The invariant set $M_{(2)}$ of the system(2) is globally exponentially stable, i.e. if there exists a real number $\alpha.>0$ and for each Δ (0< $\Delta<+\infty$) there exists a function $K(\Delta)>0$ such that $\left\|x_0^{(2)}\right\|<\Delta$ $(0\leq(\left\|x_0^{(1)}\right\|^2+\left\|x_0^{(3)}\right\|^2)^{1/2}<+\infty$ implies $\left\|x^{(2)}(t;t_0,x_{0)}\right\|\leq$ $K(\Delta)\exp[-\alpha(t-t_0)$] for each $t\geq t_0$ [6].

Definition 2. [6]. The continuous function $\varphi:[0,a]\to R^+$ (or $\varphi:R^+\to R^+$) is said to be a function of a class K, if and only if, when:

1) $\varphi(0)=0$;

2) φ is strictly increasing in [0,a] (or R^+)

and is denoted as $\varphi\in K$.

Definition 3. [6]. A function φ belongs to KR class, if and only if, when:

1) $\varphi:R^+\to R^+$;

2) $\varphi\in K$;

3) $\lim\limits_{y\to\infty}\varphi(y)=\infty$

and is denoted as $\varphi\in KR$.

Definition 4. [6]. The functions $\varphi_1,\varphi_2\in$ K(KR) have variablles of the same ordet, if and only if, when there exist real constants $\alpha_i,\beta_i,i=1,2$ such that:

$\alpha_i\varphi_i(y)\leq\varphi_j(y)\leq\beta_i\varphi_i(y), i\neq j, i, j=1,2$.

We consider the Lyapunov vector-function V(t,x)= =(V$_1$(t,x),V$_2$(t,x)), where $V_1\left(t,x^{(1)},x^{(2)},x^{(3)}\right)$ $\in$ $C\left(R^+xG,R^+\right)$ (G = ={ $x:\left\|x^{(1,2)}\right\|\leq$ H,0< $\left\|x^{(3)}\right\|<+\infty$ }) and $V_2\left(t,x^{(1)},x^{(2)},x^{(3)}\right)\in C\left(R^+xR^n,R^+\right)$ are locally Lipschitzian on x and $V_1\left(t,0,0,x^{(3)}\right)=0$, $V_2\left(t,x^{(1)},0,x^{(3)}\right)=0$.

3. *Main results*

Considering the results in [6,7] we formulate the following theorem:

Theorem: Let the right parts of the system(2) are continuous in $R^+ xR^n$ and there exist:

1) Lyapunov vector-function $V\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) = \left(V_{1,}\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right), V_2\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right)\right)$ of the type stated above;

2) Functions $\varphi_1, \varphi_2 \in K$ and $\varphi_3, \varphi_4 \in KR$, that have variables of the same order;

3) Real constants γ_1, γ_2, N,M:

a) $N\left\|x^{(1,2)}\right\|^{\gamma_1} \leq V_1\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) \leq \varphi_1\left(\left\|x^{(1,2)}\right\|\right)$;

b) $M\left\|x^{(2)}\right\|^{\gamma_2} \leq V_2\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) \leq \varphi_3\left(\left\|x^{(2)}\right\|\right)$;

c) $\dot{V}_1\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) \leq -\varphi_2\left(\left\|x^{(1,2)}\right\|\right)$;

d) $\dot{V}_2\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) \leq -\varphi_4\left(\left\|x^{(2)}\right\|\right)$ for each $t \geq t_0$.

Then the flag of invariant sets $\{M_{(1,2)}, M_{(2)}\}$ of the system (2) is exponentially polystable in the sense of definition 1.

Proof: For the functions φ_1 and φ_2, that have variables of the same order there exist real constants α_1 and β_1, the following is fulfilled

(3) $\alpha_1\varphi_1(y) \leq \varphi_2(y) \leq \beta_1\varphi_1(y), \ (y \in [0,a])$.

From the condition 3c) of the theorem and the inequality(3) we have

(4) $\dot{V}_1\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right) \leq -\varphi_2\left(\left\|x^{(1,2)}\right\|\right) \leq -\alpha_1\varphi_1\left(\left\|x^{(1,2)}\right\|\right) \leq -\alpha_1 V_1\left(t, x^{(1)}, x^{(2)}, x^{(3)}\right)$.

The comparison equation corresponding to inequality(4) is

(5) $\dot{\nu} = -\alpha_1\nu$

and the solution of this is given in the form of

(6) $\nu(t; t_0, \nu_0) = \nu_0 \exp[-\alpha_1(t - t_0)]$.

We denote $\nu_0 = V_1\left(t_0, x_0^{(1)}, x_0^{(2)}, x_0^{(3)}\right)$.

From theorem 42.2[1] we have that

(7) $V_1\left(t_0, x_0^{(1)}, x_0^{(2)}, x_0^{(3)}\right) \leq \nu_0$

implies

(8) $V_1(t; x(t; t_0, x_0)) \leq \nu(t; t_0, \nu_0)$ for each $t \geq t_0$.

From (6),(7) and (8)we have:

(9) $V_1\left(t; x^{(1)}(t; t_0, x_0), x^{(2)}(t; t_0, x_0), x^{(3)}(t; t_0, x_0)\right) \leq V_1\left(t_0, x_0^{(1)}, x_0^{(2)}, x_0^{(3)}\right)\exp[-\alpha_1(t - t_0)]$

for each $t \geq t_0$.

Inequality (9) and condition 3a) of the theorem imply

$$\left\|x^{(1,2)}(t; t_0, x_0)\right\| \leq N^{-1/\gamma_1}\varphi_1^{1/\gamma_1}\left(\left\|x_0^{(1,2)}\right\|\right)\exp\left[\frac{-\alpha_1}{\gamma_1}(t - t_0)\right] \text{ for each } t \geq t_0.$$

We denote $\lambda = \dfrac{\alpha_1}{\gamma_1}$ and for each $\varepsilon > 0$ we choose

(10) $\delta(\varepsilon) = \varphi_1^{-1}\left(N\varepsilon^{\gamma_1}\right)$.

Then, according to (10) $\left\|x_0^{(1,2)}\right\| < \delta(\varepsilon)$ implies

$$\left\|x^{(1,2)}(t;t_0,x_0)\right\| \leq N^{-1/\gamma_1}\left(\varphi_1\left(\varphi_1^{-1}\left(N\varepsilon^{\gamma_1}\right)\right)\right)^{1/\gamma_1}\exp[-\lambda(t-t_0)] = N^{-1/\gamma_1}.N^{1/\gamma_1}\varepsilon\exp[-\lambda(t-t_0)],$$

i.e. $\left\|x^{(1,2)}(t;t_0,x_0)\right\| \leq \varepsilon\exp[-\lambda(t-t_0)]$ for each $t \geq t_0$.

With the stated above we proved that the invariant set $M_{(1,2)}$ of the system(2) is exponentially stable.

By analogy we get the estimate

(11) $\left\|x^{(2)}(t;t_0,x_0)\right\| \leq M^{-1/\gamma_2}\varphi_3^{1/\gamma_2}\left(\left\|x_0^{(2)}\right\|\right)\exp\left[\dfrac{-\alpha_3}{\gamma_2}(t-t_0)\right]$ for each $t \geq t_0$.

We denote $\alpha = \dfrac{\alpha_3}{\gamma_2}$ and for each Δ ($0 < \Delta < +\infty$) and $K(\Delta) = M^{-1/\gamma_2}\varphi_3^{1/\gamma_2}(\Delta)$.

Estimate (11) and $\left\|x_0^{(2)}\right\| < \Delta$ imply $\left\|x^{(2)}(t;t_0,x_0)\right\| \leq K(\Delta)\exp[-\alpha(t-t_0)]$ for each $t \geq t_0$.

With the stated above we prove that the invariant set $M_{(2)}$ of the system(2) is globally exponentially stable.

Hence the flag of invariant sets $\{M_{(1,2)}, M_{(2)}\}$ of the system(2) is exponentially polystable.

Example: Let the system

(12) $$\begin{cases} \dot{y} = -y + z_1 - 2z_2 \\ \dot{z}_1 = 4y + z_1 \\ \dot{z}_2 = 2y + z_1 - z_2 \end{cases}$$

is given.

We denote $x^{(1)} = y$, $x^{(2)} = z_1 - 2z_2$, $x^{(3)} = z_1$ and we get

(13) $$\begin{cases} \dot{x}^{(1)} = -x^{(1)} + x^{(2)} = X_1 \\ \dot{x}^{(2)} = -x^{(2)} = X_2 \\ \dot{x}^{(3)} = 4x^{(1)} + x^{(3)} = X_3 \end{cases}$$

The right parts of the system(13) are defined in the regions

G= $\left\{\left(x^{(1)}, x^{(2)}, x^{(3)}\right) : \left(\left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2\right)^{1/2} \leq H, 0 \leq \left\|x^{(3)}\right\| < +\infty\right\}$

and $G^* = \left\{\left(x^{(1)}, x^{(2)}, x^{(3)}\right) : \left\|x^{(2)}\right\| \leq H, 0 \leq \left(\left(x^{(1)}\right)^2 + \left(x^{(3)}\right)^2\right)^{1/2} < +\infty\right\}$

It is obvious that $X_1\left(t,0,0,x^{(3)}\right) \equiv 0$, $X_2\left(t,x^{(1)},0,x^{(3)}\right) \equiv 0$.

Consequently the system(13) has a flag of invariant sets $\{M_{(1,2)}, M_{(2)}\}$,where $M_{(1,2)} = \left\{\left(x^{(1)}, x^{(2)}, x^{(3)}\right) : x^{(1)} = x^{(2)} = 0\right\}$,

$M_{(2)} = \{(x^{(1)}, x^{(2)}, x^{(3)}) : x^{(2)} = 0\}$

We shall show that the flag of invariant sets $\{M_{(1,2)}, M_{(2)}\}$ is exponentially polystable in the sense of definition 1.

We use $V = (V_1, V_2)$, where $V_1(t, x^{(1)}, x^{(2)}, x^{(3)}) = (x^{(1)})^2 + (x^{(2)})^2$, $V_2(t, x^{(1)}, x^{(2)}, x^{(3)}) = (x^{(2)})^2$.

$V_1(t, 0, 0, x^{(3)}) = 0$, $V_2(t, x^{(1)}, 0, x^{(3)}) = 0$ and hence the chosen Lyapunov vector-function satisfies the conditions of the theorem.

Let $\varphi_1(\|x^{(1,2)}\|) = \|x^{(1,2)}\|^2 = (x^{(1)})^2 + (x^{(2)})^2 = \varphi_2(\|x^{(1,2)}\|)$

$\varphi_3(\|x^{(2)}\|) = \|x^{(2)}\|^2 = (x^{(2)})^2$

$\varphi_4(\|x^{(2)}\|) = 2\|x^{(2)}\|^2 = 2(x^{(2)})^2$

it is obvious that $\varphi_1, \varphi_2 \in$ K and they have variables of the same order; $\varphi_3, \varphi_4 \in$ KR and they have variables of the same order.

Becouse

$$\dot{V}_1 = -(x^{(1)} - x^{(2)})^2 - ((x^{(1)})^2 + (x^{(2)})^2) \le -\varphi_2(\|x^{(1,2)}\|),$$

$$V_1 = \varphi_1(\|x^{(1,2)}\|),$$

$$\dot{V}_2 = -2(x^{(2)})^2 = -\varphi_4(\|x^{(2)}\|),$$

$$V_2 = (x^{(2)})^2 = \varphi_3(\|x^{(2)}\|),$$

for N=M=1, $\gamma_1 = \gamma_2 = 2$ and for the chosen functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ the conditions of the theorem are satisfied. Consequently the flag of invariant sets $\{M_{(1,2)}, M_{(2)}\}$ of sysrem(13) is exponentially polystable in the sense of definition 1.

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