ЮБИЛЕЙНА НАУЧНА СЕСИЯ – 30 години ФМИ
ПУ "Паисий Хилендарски", Пловдив, 3-4.11.2000

NUMERICAL METHOD WITH ORDER *t* FOR SOLVING SYSTEM NONLINEAR EQUATIONS

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In this paper an iteration procedure for receiving methods for solving system nonlinear equations is described. A convergence of iteration with order *t* is proved. The presented method is a generalization of the Newton's classical method of solving system nonlinear equations. Numerical examples are provided.

2000 Mathematics Subject Classification: 65H10 Systems of equations.

In this paper we will consider the problem of solving equations in R_n. The problem of solving operator equations is explored by many authors (Kantorovich and Akilov [3], Ortega и Rheinboldt [2], Collatz [1]), but in practice the method of Newton–Kantorovich is most commonly applied and it is developed for the most frequently occurring case of equations in R_n (system of n nonlinear algebraic or transcendent equations with n variables). Here we will construct iteration procedures (processes) just for this case, which is most important in practice. To these procedures belongs classical Newton's method. Here the Obreshkoff's iteration formula [4] is also generalized in R_n. General iteration process, which possesses order of convergence, t is constructed. From this process, Newton's method ($t = 2$) and Obreshkoff's method ($t = 3$) are received as particular cases. The rate of convergence of obtained methods is proved and its order of convergence is substantiated. The used technique is based on generalized Taylor's formula. The results of numerical experiments when we apply methods of order II, III, IV and V are shown and they confirm the presented theory.

Let the system of equations

(1) $$f_i(\vec{x}) = 0, i = 1,2,...,n$$

be given. Supposed that f_i and the partial derivatives of these functions of sufficiently high order are continuous in the neighborhood of solution $\vec{\xi}(\xi_1, \xi_2, ..., \xi_n)$. Using Taylor's formula we receive

(2) $$0 = f_i(\vec{\xi}) = f_i(\vec{x}_{(k)}) + f_{ij}^{(1)}(\xi^i - x_{(k)}^j) + O_i(\varepsilon^2),$$

where $\vec{x}_{(k)}(x_{(k)}^1, x_{(k)}^2, ..., x_{(k)}^n)$ is a vector of k^{th} approximation to solution, $h^j = \xi^j - x_{(k)}^j$ and $\varepsilon = \max_j |h^j|$.

With $f_{ij}^{(1)} = \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{\vec{x}_{(k)}}$ we denote the matrix of first partial derivatives of functions f_i.

We will introduce norm of multidimensional matrix by the following way:

$$\left\|a_{ijl\ldots s}\right\| = \max_i \sum_j \sum_l \ldots \sum_s \left|a_{ijl\ldots s}\right| .$$

Let us now consider

$$\left|a_{ijl\ldots s} a^j b^l \ldots c^s\right| \le \left\|a^j\right\|\left\|b^l\right\|\ldots\left\|c^s\right\|\sum_j \sum_l \ldots \sum_s \left|a_{ijl\ldots s}\right|$$

$$\le \left\|a^j\right\|\left\|b^l\right\|\ldots\left\|c^s\right\| \max_i \sum_j \sum_l \ldots \sum_s \left|a_{ijl\ldots s}\right| \le \left\|a_{ijl\ldots s}\right\|\left\|a^j\right\|\left\|b^l\right\|\ldots\left\|c^s\right\|.$$

By this way we find that

(3)
$$\left\|a_{ijl\ldots s} a^j b^l \ldots c^s\right\| \le \left\|a_{ijl\ldots s}\right\|\left\|a^j\right\|\left\|b^l\right\|\ldots\left\|c^s\right\| .$$

As a result from (3) we receive $\left\|\vec{h}\right\| = \max_j \left|h^j\right|$ and $\left\|f^{(1)}\right\| = \max_i \sum_{j=1}^{n} \left|f_{ij}^{(1)}\right|$. Supposed that $\left\{f_{i\ j}^{(1)}\right\}$ is a non-singular matrix. We denote with $\left\{f_{(1)}^{s\ l}\right\}$ reciprocal matrix of $\left\{f_{ij}^{(1)}\right\}$. We will use Einstein rule, according to which if in one product there is identical superscript and subscript index this means that we have the sum at this index from 1 to n. Using our denotes this two conditions follow

(4)
$$f_{ij}^{(1)} f_{(1)}^{j\quad l} = \delta_i^l$$

and

(5)
$$f_{(1)}^{sj} f_{jl}^{(1)} = \delta_l^s ,$$

where δ_i^l is a symbol of Kronecker $\delta_i^l = \begin{cases} 0, & i \ne l \\ 1, & i = l \end{cases}$.

We multiply left and right side of (2) by $f_{(1)}^{l\ i}$ and we recieve:
$0 = f_{(1)}^{li} f_i\left(\vec{x}_{(k)}\right) + f_{(1)}^{li} f_{ij}^{(1)}\left(\xi^i - x_{(k)}^j\right) + f_{(1)}^{li} O_i\left(\varepsilon^2\right)$. From (5) it follows that $0 = f_{(1)}^{li} f_i\left(\vec{x}_{(k)}\right) + \delta_j^l h^j + f_{(1)}^{li} O_i\left(\varepsilon^2\right)$.

We denote $f_{(1)}^{li} O_i\left(\varepsilon^2\right)$ with $O^l\left(\varepsilon^2\right)$ and solve the last equations with respect of h^l. In this way we receive

(6)
$$h^l = -f_{(1)}^{li} f_i + O^l\left(\varepsilon^2\right).$$

Let us denote

(7)
$$H_1^j = -f_{(1)}^{jl} f_l .$$

From (6) and (7) it follows that

(8)
$$h^l = H_1^l + O^l\left(\varepsilon^2\right).$$

It is known, that

$$O_i\left(\varepsilon^2\right) = \frac{1}{2}\left(\frac{\partial^2 f_i}{\partial x^j \partial x^l}\right)_{\vec{x}_{(k)} + \theta\left(\vec{\xi} - \vec{x}_{(k)}\right)} h^j h^l ,$$

where $0 < \theta < 1$. Because of the continuity of partial derivatives of functions f_i we can concede that positive number M_2 exists, such that, in sufficiently small neighborhood of solution $\vec{\xi}(\xi_1,\xi_2,...,\xi_n)$ inequalities $\left|\frac{\partial^2 f_i}{\partial x^j \partial x^l}\right| < M_2$ are fulfilled for $\forall j$ and $\forall l$. As a result we receive the following upper bound

$$\left|O_i(\varepsilon^2)\right| \le \frac{1}{2}\left|\frac{\partial^2 f_i}{\partial x^j \partial x^l}\right|\left|h^j\right|\left|h^l\right| \le \frac{1}{2} n^2 M_2 \varepsilon^2$$

From the condition that the matrix $\left\{f_i{}^{(1)}_j\right\}$ is non-singular it follows that $\left\| f_{(1)}^{i\ l} \right\|$ is a positive number.

The estimate (9) is valid

(9) $$\left\|O^i(\varepsilon^2)\right\| \le \left\| f_{(1)}^{i\ j} \right\| \left\|O_j(\varepsilon^2)\right\|.$$

Newton's method is received when

(10) $$x^s_{(k+1)} = x^s_{(k)} + H^s_1 .$$

After using relation (10) we obtain

$$\xi^s - x^s_{(k+1)} = \xi^s - x^s_{(k)} - H^s_1 = h^s - H^s_1 = O^s(\varepsilon^2).$$

Consequently, the estimates are fulfilled

$$\left\|\xi^s - x^s_{(k+1)}\right\| = \left\|O^s(\varepsilon^2)\right\| \le \frac{1}{2} n^2 M_2 \varepsilon^2 \left\| f_{(1)}^{i\ l} \right\|.$$

From this fact it follows that the method (10) is of second order with respect of ε.

If in Taylor's formula we take into account the terms of second degree with respect of h^i we receive

(11) $$0 = f_i + f_{ij}^{(1)} h^j + \frac{1}{2} f_{ijl}^{(1)} h^j h^l + \overline{O}_i(\varepsilon^3),$$

where

$$\overline{O}_i(\varepsilon^3) = \frac{1}{6} \left.\frac{\partial^3 f_i}{\partial x^l \partial x^j \partial x^s}\right|_{\vec{x}_{(k)}+\theta(\vec{\xi}-\vec{x}_{(k)})} h^l h^j h^s .$$

It is evident that $\left|\overline{O}_i(\varepsilon^3)\right| = \frac{1}{6}\left|\frac{\partial^3 f_i}{\partial x^l \partial x^j \partial x^s}\right|\left|h^l\right|\left|h^j\right|\left|h^s\right|$. As it was mentioned above the constant M_3 exists such that $\left|\frac{\partial^3 f_i}{\partial x^l \partial x^j \partial x^s}\right| < M_3$ for every l, j and s. Hence, $\left\|\overline{O}_i(\varepsilon^3)\right\| \le \frac{1}{6} n^3 M_3 \varepsilon^3$.

From (8) and (11) with made supposition we found

(12) $$0 = f_i + \left(f_{ij}^{(1)} + \frac{1}{2} f_{ijl}^{(1)} H^l_1 \right) h^j + O_i(\varepsilon^3),$$

where for brevity we substitute $O_i(\varepsilon^3) = \overline{O}_i(\varepsilon^3) + \frac{1}{2} f_{ijl}^{(1)} h^j O^l(\varepsilon^2)$.

Let $\left\{f_{(2)}^{ij}\right\}$ be reciprocal matrix of the matrix $\left\{f_{ij}^{(2)}\right\} = \left\{ f_{ij}^{(1)} + \frac{1}{2} f_{ijl}^{(1)} H^l_1 \right\}$. The matrix $\left\{f_{(2)}^{ij}\right\}$ exists because in sufficiently small neighborhood of solution the matrix $\left\{f_{ij}^{(2)}\right\}$ is arbitrarily near to $\left\{f_{ij}^{(1)}\right\}$ and consequently is a non-singular matrix. After replacing in (12) we receive

(13)
$$0 = f_i + f_{ij}^{(2)} h^j + O_i\left(\varepsilon^3\right).$$

Analogously to previous considerations, we solve (13) with respect of h^j and obtain

(14)
$$h^l = H_2^l + O^l\left(\varepsilon^3\right),$$

where we denote $H_2^l = -f_{(2)}^{li} f_i$ and $O^l\left(\varepsilon^3\right) = f_{(2)}^{li} O_i\left(\varepsilon^3\right)$. Thereby analogue of Obreshkoff's formula [4] for system of nonlinear equations is received

(15)
$$x_{(k+1)}^s = x_{(k)}^s + H_2^s .$$

As in Newton's method, we found

$$\xi^s - x_{(k)}^s - H_2^s = h^s - H_2^s = \xi^s - x_{(k+1)}^s = O^s\left(\varepsilon^3\right).$$

From expression of $O_i\left(\varepsilon^3\right)$ we receive the following upper bound

$$\left\|O_i\left(\varepsilon^3\right)\right\| \le \left\|\overline{O}_i\left(\varepsilon^3\right)\right\| + \frac{1}{2}\left\| f_{ijl}^{(1)} h^j \right\| \left\| O^l\left(\varepsilon^2\right)\right\| .$$

On the other hand, we have

$$\sum_j \left| f_{ijl}^{(1)} h^j \right| \le \max_i \sum_k \left| f_{ijl}^{(1)} \right| \left| h^j \right| \le \varepsilon\, n M_2 .$$

Consequently, the estimate $\left\| f_{ijl}^{(1)} h^j \right\| \le \varepsilon\, n M_2$ is fulfilled.

From the obtained bounds it follows that

$$\left\|O_i\left(\varepsilon^3\right)\right\| \le \frac{1}{6} n^3 M_3 \varepsilon^3 + \frac{1}{2} \varepsilon n M_2 \frac{1}{2} n^2 M_2 \varepsilon^2 = \left[\frac{1}{6} n^3 M_3 + \frac{1}{4} n^3 \left(M_2\right)^2\right] \varepsilon^3 \overset{\text{def}}{=\!=} M_3^* \varepsilon^3$$

and

(16)
$$\left\|\xi^s - x_{(k+1)}^s\right\| = \left\|O^s\left(\varepsilon^3\right)\right\| \le \left\| f_{(2)}^{ij} \right\| M_3^* \varepsilon^3 ,$$

which shows that method (15) is of third order with respect of ε.

In the case when in Taylor's formula we take into account the terms of the third order with respect of h^s, we receive

(17)
$$0 = f_i + f_{ij}^{(1)} h^j + \frac{1}{2} f_{ijs}^{(1)} h^j h^s + \frac{1}{6} f_{ijsl}^{(1)} h^j h^s h^l + \overline{O}_i\left(\varepsilon^4\right).$$

After using (14), the system of equations (17) comes to

$$0 = f_i + \left(f_{ij}^{(1)} + \frac{1}{2} f_{ijs}^{(1)} H_2^s + \frac{1}{6} f_{ijsl}^{(1)} H_2^s H_2^l \right) h^j$$

$$+ \frac{1}{2} f_{ijs}^{(1)} O^s\left(\varepsilon^3\right) h^j + \frac{1}{6} f_{ijsl}^{(1)} \left[O^s\left(\varepsilon^3\right) H_2^l + O^l\left(\varepsilon^3\right) H_2^s + O^s\left(\varepsilon^3\right) O^l\left(\varepsilon^3\right)\right] h^j + \overline{O}_i\left(\varepsilon^4\right).$$

For brevity we introduce a vector $O_i\left(\varepsilon^4\right)$ and the matrix $f_{i\ j}^{(3)}$ as follows:

$$O_i\left(\varepsilon^4\right) = \frac{1}{2} f_{ijs}^{(1)} O^s\left(\varepsilon^3\right) h^j + \frac{1}{6} f_{ijsl}^{(1)} O^s\left(\varepsilon^3\right) H_2^l h^j + \frac{1}{6} f_{ijsl}^{(1)} O^l\left(\varepsilon^3\right) H_2^s h^j$$

$$+ \frac{1}{6} f_{ijsl}^{(1)} O^s\left(\varepsilon^3\right) O^l\left(\varepsilon^3\right) h^j + \overline{O}_i\left(\varepsilon^4\right),$$

(18) $$f_{ij}^{(3)} = f_{ij}^{(1)} + \frac{1}{2} f_{ijs}^{(1)} H_2^s + \frac{1}{6} f_{ijsl}^{(1)} H_2^s H_2^l .$$

Contacted writing of system is

(19) $$0 = f_i + f_{ij}^{(3)} h^j + O_i\left(\varepsilon^4\right).$$

We solve (19) with respect of h^j and receive

(20) $$h^j = -f_{(3)}^{js} f_s - f_{(3)}^{js} O_s\left(\varepsilon^4\right).$$

Here $\left\{f_{(3)}^{js}\right\}$ is a reciprocal matrix of the matrix $\left\{f_{sl}^{(3)}\right\}$. In this way the system (20) receives the form

(21) $$h^j = H_3^j + O^j\left(\varepsilon^4\right),$$

where $O^j\left(\varepsilon^4\right) = -f_{(3)}^{j\ s} O_s\left(\varepsilon^4\right)$ and $H_3^j = -f_{(3)}^{j\ s} f_s$.

The new iteration method for solving system of nonlinear equations is

(22) $$x_{(k+1)}^s = x_{(k)}^s + H_3^s .$$

We constitute

$$\xi^s - x_{(k+1)}^s = \xi^s - x_{(k)}^s - H_3^s = h^s - H_3^s = O^s\left(\varepsilon^4\right).$$

The following bounds are valid

$$\left\|O_i\left(\varepsilon^4\right)\right\| \le \frac{1}{2}\left\|f_{ijs}^{(1)}\right\|\left\|O^s\left(\varepsilon^3\right)\right\|\left\|h^j\right\| + \frac{1}{6}\left\|f_{ijsl}^{(1)}\right\|\left\|O^s\left(\varepsilon^3\right)\right\|\left\|H_2^l\right\|\left\|h^j\right\|$$

$$+\frac{1}{6}\left\|f_{ijsl}^{(1)}\right\|\left\|O^l\left(\varepsilon^3\right)\right\|\left\|H_2^s\right\|\left\|h^j\right\| + \frac{1}{6}\left\|f_{ijsl}^{(1)}\right\|\left\|O^s\left(\varepsilon^3\right)\right\|\left\|O^l\left(\varepsilon^3\right)\right\|\left\|h^j\right\| + \left\|\overline{O}_i\left(\varepsilon^4\right)\right\|$$

$$\le \frac{1}{2}\left\|f_{ijs}^{(1)}\right\| M_3^* \varepsilon^4 + \frac{1}{6}\left\|f_{ijsl}^{(1)}\right\| M_3^* \varepsilon^4 \left\|H_2^l\right\| + \frac{1}{6}\left\|f_{ijsl}^{(1)}\right\|\left\|H_2^s\right\| M_3^* \varepsilon^4$$

$$+\frac{1}{6}\left\|f_{ijsl}^{(1)}\right\|\left(M_3^*\right)^2 \varepsilon^7 + \left\|\overline{O}_i\left(\varepsilon^4\right)\right\|$$

and

$$\left\|\overline{O}_i\left(\varepsilon^4\right)\right\| = \left\|\frac{1}{4!} f_{ijslm}^{(1)} h^j h^s h^l h^m\right\| \le \frac{1}{4!}\left\|f_{ijslm}^{(1)}\right\| \varepsilon^4 ,$$

from which it follows that

(23) $$\left\|O_i\left(\varepsilon^4\right)\right\| \le M_4^* \varepsilon^4 ,$$
$$\left\|\xi^s - x_{(k+1)}^s\right\| = \left\|O^s\left(\varepsilon^4\right)\right\| \le \left\|f_{(3)}^{js}\right\|\left\|O_s\left(\varepsilon^4\right)\right\| \le \left\|f_{(3)}^{js}\right\| M_4^* \varepsilon^4 ,$$

where M_4^* is a positive constant, the existence of which is guaranteed by sufficient smoothness of the functions. The inequalities (23) guarantee, that the new method (22) has fourth order of convergence with respect to ε.

We will generalize both Newton's and Obreshkoff's method by exploring a general case when in Taylor's formula all the terms up to degree $t-1$ with respect of h^s will be included. Taylor's formula is of the form

(24) $$0 = f_i + f_{ii_1}^{(1)} h^{i_1} + \frac{1}{2!} f_{ii_1i_2}^{(1)} h^{i_1} h^{i_2} + \frac{1}{3!} f_{ii_1i_2i_3}^{(1)} h^{i_1} h^{i_2} h^{i_3} + \ldots + \frac{1}{(t-1)!} f_{ii_1i_2\ldots i_{t-1}}^{(1)} h^{i_1} h^{i_2} \ldots h^{i_{t-1}} + \overline{O}\left(\varepsilon^t\right).$$

Following the scheme by which Newton's and Obreshkoff's iteration methods were developed we suppose that in the above consideration we have determined

(25)
$$h^j = H^j_{t-2} + O^j\left(\varepsilon^{t-1}\right)$$

and

(26)
$$\left\| O_i\left(\varepsilon^{t-1}\right) \right\| \le M^*_{t-1}\varepsilon^{t-1}.$$

Then, after replacing h^j from (25) in (24) we receive

(27)
$$0 = f_i + \left(f^{(1)}_{ii_1} + \frac{1}{2!} f^{(1)}_{ii_1i_2} H^{i_2}_{t-2} + \frac{1}{3!} f^{(1)}_{ii_1i_2i_3} H^{i_2}_{t-2} H^{i_3}_{t-2} + \ldots + \frac{1}{(t-1)!} f^{(1)}_{ii_1i_2\ldots i_{t-1}} H^{i_2}_{t-2} H^{i_3}_{t-2} \ldots H^{i_{t-1}}_{t-2} \right) h^{i_1}$$
$$+ \frac{1}{2!} f^{(1)}_{ii_1i_2} O^{i_2}\left(\varepsilon^{t-1}\right) h^{i_1} + \frac{1}{3!} f^{(1)}_{ii_1i_2i_3} \left(O^{i_2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2} + O^{i_3}\left(\varepsilon^{t-1}\right) H^{i_2}_{t-2} + O^{i_2}\left(\varepsilon^{t-1}\right) O^{i_3}\left(\varepsilon^{t-1}\right) \right) h^{i_1}$$
$$+ \ldots + \frac{1}{(t-1)!} f^{(1)}_{ii_1i_2\ldots i_{t-1}} \left[O^{i_2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2} H^{i_4}_{t-2} \ldots H^{i_{t-1}}_{t-2} + \ldots + H^{i_2}_{t-2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2}\left(\varepsilon^{t-1}\right) \ldots \right.$$
$$\left. \ldots O^{i_{t-1}}\left(\varepsilon^{t-1}\right) + O^{i_2}\left(\varepsilon^{t-1}\right) O^{i_3}\left(\varepsilon^{t-1}\right) \ldots \right] h^{i_1} + \overline{O}_i\left(\varepsilon^t\right).$$

The system (27) we write in the form

(28)
$$0 = f_i + f^{(t)}_{ii_1} h^{i_1} + O_i\left(\varepsilon^t\right),$$

where we substitute

$$f^{(t)}_{ii_1} = f^{(1)}_{ii_1} + \frac{1}{2!} f^{(1)}_{ii_1i_2} H^{i_2}_{t-2} + \ldots + \frac{1}{(t-1)!} f^{(1)}_{ii_1i_2\ldots i_{t-1}} H^{i_2}_{t-2} H^{i_3}_{t-2} \ldots H^{i_{t-1}}_{t-2}$$

and

$$O_i\left(\varepsilon^t\right) = \frac{1}{2!} f^{(1)}_{ii_1i_2} O^{i_2}\left(\varepsilon^{t-1}\right) h^{i_1}$$
$$+ \frac{1}{3!} f^{(1)}_{ii_1i_2i_3} \left[O^{i_2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2} + O^{i_3}\left(\varepsilon^{t-1}\right) H^{i_2}_{t-2} + O^{i_2}\left(\varepsilon^{t-1}\right) O^{i_3}\left(\varepsilon^{t-1}\right) \right] h^{i_1}$$
$$+ \ldots + \frac{1}{(t-1)!} f^{(1)}_{ii_1i_2\ldots i_{t-1}} \left[O^{i_2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2} H^{i_4}_{t-2} \ldots H^{i_{t-1}}_{t-2} \right.$$
$$\left. + \ldots + H^{i_2}_{t-2}\left(\varepsilon^{t-1}\right) H^{i_3}_{t-2}\left(\varepsilon^{t-1}\right) \ldots O^{i_{t-1}}\left(\varepsilon^{t-1}\right) + \ldots \right] h^{i_1} + \overline{O}_i\left(\varepsilon^t\right).$$

The elements of reciprocal matrix of $\left\{ f_{i\ i_1}^{(t)} \right\}$ we denote with $f_{(t)}^{i\ i_1}$. From the system (28) we determine h^{i_1}

(29)
$$h^{i_1} = -f^{i_1 s}_{(t)} f_s + f^{i_1 s}_{(t)} O_s\left(\varepsilon^t\right).$$

For brevity we substitute $f^{i_1 s}_{(t)} O_s\left(\varepsilon^t\right) = O^{i_1}\left(\varepsilon^t\right)$ and $H^{i_1}_{t-1} = -f^{is}_{(t)} f_s$.

We form the following iteration method:

(30)
$$x^s_{(k+1)} = x^s_{(k)} + H^s_{t-1}$$

for solving of nonlinear system.

Let us form the expression

$$\xi^s - x^s_{(k)} - H^s_t = \xi^s - x^s_{(k+1)} = h^s - H^s_t = O^s\left(\varepsilon^t\right).$$

Analogously to the above mentioned considerations we have

$$\begin{aligned}\left\|O_i\left(\varepsilon^t\right)\right\| &\le \frac{1}{2!}\left\|f^{(1)}_{i\,i_1\,i_2}\right\|\left\|O^{i_2}\left(\varepsilon^{t-1}\right)\right\|\left\|h^{i_1}\right\| \\ &+\frac{1}{3!}\left[\left\|f^{(1)}_{i\,i_1\,i_2\,i_3}\right\|\left\|O^{i_2}\left(\varepsilon^{t-1}\right)\right\|\left\|h^{i_1}\right\|+\left\|f^{(1)}_{i\,i_1 i_2\,i_3}\right\|\left\|O^{i_3}\left(\varepsilon^{t-1}\right)\right\|\left\|h^{i_1}\right\|+\dots\right] \\ &+\dots+\frac{1}{(t-1)!}\left[\left\|f^{(1)}_{i\,i_1\,i_2\,\dots\,i_{t-1}}\right\|\left\|O^{i_2}\left(\varepsilon^{t-1}\right)\right\|\left\|H^{i_3}_{t-1}\right\|\dots\left\|H^{i_{t-1}}_{t-1}\right\|\right. \\ &\left.+\dots+\left\|f^{(1)}_{i\,i_1\,i_2\,\dots\,i_{t-1}}\right\|\left\|H^{i_2}_{t-1}\right\|\left\|H^{i_3}_{t-1}\right\|\dots\left\|O^{i_{t-1}}\left(\varepsilon^{t-1}\right)\right\|+\dots\right]\left\|h^{i_1}\right\|+\left\|\overline{O}\left(\varepsilon^t\right)\right\|\end{aligned}$$

and

$$\left\|\overline{O}_i\left(\varepsilon^t\right)\right\| = \left\|f^{(1)}_{ii_1i_2\dots i_t}h^{i_1}h^{i_2}\dots h^{i_t}\right\| \le \left\|f^{(1)}_{ii_1i_2\dots i_t}\right\|\left\|h^{i_1}\right\|\left\|h^{i_2}\right\|\dots\left\|h^{i_t}\right\| \le \left\|f^{(1)}_{ii_1i_2\dots i_t}\right\|\varepsilon^t,$$

where all addends which contained multipliers ε^s are dropped, when $s > t$.

Taking into account the fact that functions f_i are sufficiently smooth and that in above estimates all addends are of order $O_i\left(\varepsilon^t\right)$.

We conclude that the following upper bound is valid

(31)
$$\left\|O_i\left(\varepsilon^t\right)\right\| \le M_t^*\varepsilon^t,$$

where M_t^* is a positive constant.

Using (31) we receive the inequalities

$$\left\|\xi^s - x^s_{(k+1)}\right\| \le \left\|O^s\left(\varepsilon^t\right)\right\| \le M_t^*\varepsilon^t\left\|f^{i_1\ \ l}_{(t)}\right\|,$$

which proves that the order of convergence of this method is t.

Numerical example. For the system $\begin{vmatrix}3x_1^2x_2 + x_2^2 = 1\\ x_1^4 + x_1x_2^3 = 1\end{vmatrix}$, after applying the described above technique for receiving iteration methods at initial approximations $x_1^{[0]} = 2$ and $x_2^{[0]} = -1$, we receive the following numerical results (the correct digits in the results are marked by bold type).

Method (10) – Newton's method (II order of convergence)		
k	$x_1^{[k]}$	$x_2^{[k]}$
0	2.000000000000000000	–1.000000000000000000
1	1.471204188481675390	–0.434554973821989529
2	1.160971103732131220	–0.000211512078262731
3	1.030491163618779090	0.247285062098385618
4	**0.99**5486960519633108	**0.30**2874141673445504
5	**0.9927**94407241188532	**0.3064**22485001680910
6	**0.99277999**5253887578	**0.306440446**016981499
7	**0.992779994851123249**	**0.30644044651102043**1
8	**0.992779994851123249**	**0.306440446511020432**
9	**0.992779994851123249**	**0.306440446511020432**

Method (15) – Obreshkoff's method (III order of convergence)		
k	$x_1^{[k]}$	$x_2^{[k]}$
0	2.000000000000000000	–1.000000000000000000
1	1.236361502136902590	–0.102010783027205119
2	1.016236675279352840	0.283124619837572002
3	**0.992**806803517828091	**0.30**6410483449974681
4	**0.992779994**851170731	**0.306440446**510967770
5	**0.992779994851123249**	**0.306440446511020432**
6	**0.992779994851123249**	**0.306440446511020432**

Method (22) (IV order of convergence)		
k	$x_1^{[k]}$	$x_2^{[k]}$
0	2.000000000000000000	–1.000000000000000000
1	1.132550738861533230	0.023572314322562824
2	**0.99**4110525451864892	**0.30**3989504948906135
3	**0.992779944**876562587	**0.306440446**474358190
4	**0.992779994851123249**	**0.306440446511020432**
5	**0.992779994851123249**	**0.306440446511020432**

Method (30) at $t = 5$ (V order of convergence)		
k	$x_1^{[k]}$	$x_2^{[k]}$
0	2.000000000000000000	–1.000000000000000000
1	1.082281042482679530	0.123366196386319406
2	**0.992**837748938471569	**0.306**361894605406281
3	**0.992779994851123249**	**0.306440446511020432**
4	**0.992779994851123249**	**0.306440446511020432**

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