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ON TRANSFORMATION OF AFFINED COHERENCES

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Assume there is a mutual simple and fluxionable correspondence existing between the points of two spaces of affined coherence, namely ^{1}**An** and ^{2}**An**. When the validity of the condition $\nabla_{[k}v_{i]}=0$ for a random co-vector from one of the spaces premises the validity of the condition for the corresponding co-vector in the other space, we say there is a GMC-correspondence (Generalized Metric Chebyshev Correspondence) between the two spaces. Certain characteristics of the above-mentioned correspondence have been found. Research has also been done on other GMC-correspondences that are projective or conjugated with respect to a bivalent tensor.

1. Basic facts.

Suppose there is a mutual simple and fluxionable correspondence existing between two separate spaces of affined coherence namely ^{1}**An** and ^{2}**An**. A joint coordinate system can be introduced in which the corresponding points have equal coordinates.

Let us note the coefficients of coherence of the spaces ^{1}**An** and ^{2}**An** respectively ${}^1\Gamma_{is}^k$ and ${}^2\Gamma_{is}^k$. The transition from one coherence into the other is considered a transformation of the coherences or, otherwise, a transformation of the Parallel Transfer Law.

If $\mathbf{v}^i$ is a random vector field, we have the following expressions for the co-variant derivates in the two coherences:

$$(1)\qquad {}^1\nabla_k v^i = \partial_k v^i + {}^1\Gamma_{ks}^i v^s, \; {}^2\nabla_k v^i = \partial_k v^i + {}^2\Gamma_{ks}^i v^s$$

and for their difference they are:

$$(2)\qquad {}^2\nabla_k v^i - {}^1\nabla_k v^i = T_{ks}^i v^i$$

where

$$(3)\qquad T_{ks}^i = {}^2\Gamma_{ks}^i - {}^1\Gamma_{ks}^i$$

is the tensor of affined deformation [1. p. 128].

The connection between the tensors of the curvature ${}^1R_{skm.}{}^i$ and ${}^2R_{skm.}{}^i$ respectively belonging to the spaces ^{1}**An** and ^{2}**An** is

(4) $$^2R_{skm.}{}^{i} = {}^1R_{skm.}{}^{i} + {}^1\nabla_s T^i_{km} - {}^1\nabla_k T^i_{sm} + T^i_{sp}T^p_{km} - T^i_{kp}T^p_{sm} + 2\,{}^1S^p_{sk}T^i_{pm}$$

where ${}^1S^p_{sk}$ denies the tensor of torsion of the space **^{1}An** [1. p. 130].

The independent vector fields $\underset{\alpha}{v}^i$ (α=1,2…n) in the spaces **^{1}An** and **^{2}An** determine the network **($v_1,v_2…v_n$)** in these spaces. The mutual vectors $\overset{\alpha}{v}_i$ of the vectors $\underset{\alpha}{v}^i$ (α=1,2…n) are definitions with the help of the following equation

(5) $$\underset{\alpha}{v}^i \overset{\beta}{v}_i = \delta_\alpha^\beta \quad , \qquad (\underset{\alpha}{v}^i \overset{\alpha}{v}_k = \delta^i_k)$$

Following [2] and [3] we shall call the network ($v_1,v_2…v_n$) a generalized metric network of Chebyshev, if for α=1,2…n. The following condition is satisfied:

(6) $$^a\nabla_{[k} \overset{\alpha}{v}_{i]} = 0$$

In case of space **^{a}An** being a two-or-three-dimensional Rumanov's space or **n**-dimensional space of the condition (6) characterizes the generalized metric nets of Chebyshev, studied in [4], [5] and [2] respectively.

A. P. Norden introduced the following definitions and proved the statements. [1].

Definition 1: A transformation of coherences which leaves unaltered the geodesic lines in spaces **^{1}An** and **^{2}An** is known as projective, the spaces are considered projective of each other. [1. p. 165].

Statement 1: Spaces **^{1}An** and **^{2}An** are projective between each other then and only then when the tensor of affined deformation satisfies the condition:

(7) $$\delta^{[i}_{(m}T^{s]}_{k)} = 0$$

Statement 2: If there is aprojective correspondence between two spaces having a symmetric affined coherences, the tensor of affined deformation satisfies the condition [1. p. 166].

(8) $$T^k_{is} = T^{km}_{\cdot\,\cdot\, is}\, p_m$$

where

(9) $$p_m = \frac{1}{n+1} T^{s}_{sm}$$

(10) $$T^{km}_{\;\cdot\,\cdot\,is} = \delta^k_i \delta^m_s + \delta^k_s \delta^m_i$$

The properties of the tensor (10) have been studied in [1. p. 166].

Let spaces ^{1}**An** and ^{2}**An** contain a non-singular symmetric tensor $\mathbf{b_{is}}$.

Definition 2: Vectors $v^i e^1 An$ and $w^s e^2 An$ are conjugated with respect to the tensor $\mathbf{b_{is}}$ if the condition [1. p. 173] is satisfied:

(11) $$b_{is} v^i w^s = 0$$

Definition 3: The coherences $^1\Gamma^k_{is}$ and $^2\Gamma^k_{is}$ are conjugated with respect to the tensor $\mathbf{b_{is}}$, if case of parallel transfer along every curve **L** from the field $\mathbf{v^i}$ in space ^{1}**An** and field $\mathbf{w^i}$ in space ^{2}**An**, the condition (11) [1. p. 173] is satisfied.

Statement 3: Let the coherences $^1\Gamma^k_{is}$ and $^2\Gamma^k_{is}$ be conjugated with respect to the tensor $\mathbf{b_{is}}$, then the conditions [1. p. 175]

(12) $$T^s_{kj} = \tilde{b}^{is} (\delta^s_p {}^1\nabla_k b_{jm} + \frac{2}{n-1} \delta^s_j {}^1\nabla_{[p} b_{k]m})$$

are satisfied.

(13) $$T^k_{ki} = T^k_{ik}$$

where $\tilde{b}^{is} \; b_{ik} = \delta^s_k$.

The vector

(14) $$t_i = \frac{1}{n+2} T^k_{ki} = \frac{n}{(n-1)(n+2)} \tilde{b}^{sk} ({}^1\nabla_s b_{ki} - \frac{1}{n} {}^1\nabla_i b_{sk})$$

is called a Chebyshev’s vector polarity, determined by the tensor $\mathbf{b_{is}}$ [1. p.175].

The covariant derivative of the mutual vectors $\overset{\alpha}{v}_i$ of the vectors $\underset{\alpha}{v}^i$ satisfies the condition

(15) $$^2\nabla_k \overset{\alpha}{v}_i - {}^1\nabla_k \overset{\alpha}{v}_s = -T^s_{ki} \overset{\alpha}{v}_s$$

2. Transformation of affined coherences.

Theorem 1. If for the mutual co-vector of a random vector field $\underset{\alpha}{v^i} \in {}^{\mathbf{a}}\mathbf{An}$ (**a**=1,2) the conditions (6) are satisfied for (**a**=1,2), the tensor of deformation meets the following condition:

(16) $$T^s_{[ki]} = 0$$

Proof: Assume that for a random vector field $\underset{\alpha}{v^i} \in {}^{\mathbf{a}}\mathbf{An}$ the conditions ${}^1\nabla_{[k} \overset{\alpha}{v}_{i]} = {}^2\nabla_{[k} \overset{\alpha}{v}_{i]} = 0$

are satisfied. From (15) follows that $T^s_{[ki]} \overset{\alpha}{v}_s = 0$, but $\overset{\alpha}{v}_i$ is a random co-vector, consequently,

$$T^s_{[ki]} = 0$$

Conversely, let (6) and (16) be satisfied for one of the spaces ${}^{\mathbf{1}}$**An** and ${}^{\mathbf{2}}$**An**, then (15) is met identifically.

Definition 4: A transformation of coherences is called a GMC if for a random co-vector $\underset{\alpha}{v^i} \in {}^{\mathbf{a}}\mathbf{An}$ from ${}^1\nabla_{[k} \overset{\alpha}{v}_{i]}$ follows ${}^2\nabla_{[k} \overset{\alpha}{v}_{i]} = 0$ and vice versa.

Consequence 1: If there is a GMC-transformation between the spaces ${}^{\mathbf{1}}$**An** and ${}^{\mathbf{2}}$**An**, then a random generalized metric Chebyshev network from one of the spaces is transformed into a generalized metric Chebyshev network in the other space.

Proof: It follows from the definition of a generalized metric Chebyshev network [2], [7] and Theorem 1.

Consequence 2: If there is a GMC-correspondence between two spaces with affined coherence, they have equal tensors of torsion.

Proof: It follows from Theorem 1.

Consequence 3: If there is a GMC-correspondence between two spaces with affined coherence and one of them is without torsion, then the other one as well has no torsion.

Proof: It follows from Consequence 2.

Theorem 2: If there is a GMC and projective correspondence between two spaces with affined coherence, then the expressions (8) and (9) are satisfied for them.

Proof: Let there be a GMC and projective correspondence between the spaces ${}^{\mathbf{1}}$**An** and ${}^{\mathbf{2}}$**An**. From (7) for the tensor of affined deformation we obtain

(17) $$T^s_{ki} + T^s_{ik} = \frac{1}{n+1}(\delta^s_k \delta^e_i + \delta^s_i \delta^e_k)(T^m_{em} + T^m_{me})$$

Taking into account (10), (16) and (17) we find 8; where $\mathbf{p_m}$ satisfies (9).

From (4) for the tensor of the curvature of spaces ^{1}**An** and ^{2}**An** we find:

(18) $$^2R_{ski\,.}{}^{i} = {}^1R_{ski\,.}{}^{i} + {}^1\nabla_s T^{\,i}_{ki} - {}^1\nabla_k T^{\,i}_{si} + 2\,{}^1S^{\,m}_{si} T^{\,i}_{mi}$$

Consequence 4: The tensor of affined deformation of two, projective with respect to one other, spaces with affined coherence without torsion, has the same structure as the tensor of affined deformation of two projective between each other spaces with affined coherence and equal tensors of torsion.

Proof: It follows from Theorem 1 and Theorem 2.

Consequence 5: If there is a GMC and projective correspondence between spaces ^{1}**An** and ^{2}**An** and if the space ^{2}**An** is equiaffined, then the following conditions are satisfied:

(19) $$p_k = \frac{1}{n+1}(\partial_k \ln e - {}^1\Gamma^m_{km})$$

(20) $$^1R_{ski.}{}^{i} = 2\,{}^1\nabla_{[s}{}^1\Gamma^i_{k]}$$

where **e** is the density of the equiaffined space ^{2}**An**.

Proof: Assume that there is a GMC and projective correspondence between the space with affined coherence ^{1}**An** and the equiaffined ^{2}**An**.

Since the equiafined space ^{2}**An** has symmetric coherence, according to the consequence 2, ^{1}**An** also has symmetric coherence, i. e. $^1S^i_{sk} = 0$.

What we know about the equiaffined space ^{2}**An** [1. p. 151] is that

(21) $$^2\Gamma^k_{ks} = \partial_k \ln e \qquad , \qquad {}^2R_{ski.}{}^{i} = 0$$

Taking into account (3), (9) and (21), we find (19); from (3), (18) and (21) we obtain (20).

3. Special conjugated transformations of affined coherences.

Let a non-singular symmetric tensor $\mathbf{b_{is}}$ be given in spaces ^{1}**An** and ^{2}**An**.

Theorem 3: If the spaces with affined coherences are projective between each other and their coherences are conjugated with respect to the tensor $\mathbf{b_{is}}$, then $\mathbf{b_{is}}$ satisfies the condition.

(22) $$^{1}\nabla_{(k}b_{s)i} = \frac{1}{n+1}\tilde{b}^{pm}(b_{ki}\,^{1}\nabla_{(p}b_{s)m} + b_{si}\,^{1}\nabla_{(k}b_{s)m})$$

Proof: Assume **^{1}An** and **^{2}An** are projective of each other and their coherences are conjugated in terms of tensor $\mathbf{b_{is}}$. From (10), (13) and (17) we find

(23) $$T^{s}_{ki} + T^{s}_{ik} = T^{sm}_{..ki}q_{m}$$

where

(24) $$q_{m} = \frac{2}{n+1}T^{k}_{km}$$

Taking into account (12) and (24) we have:

$$\tilde{b}^{pm}(\delta^{i}_{p}\,^{1}\nabla_{k}b_{im} + \frac{2}{n-1}\delta^{s}_{i}\,^{1}\nabla_{[p}b_{k]m}) + \tilde{b}^{pm}(\delta^{s}_{p}\,^{1}\nabla_{i}b_{km} + \frac{2}{n-1}\delta^{s}_{i}\,^{1}\nabla_{[p}b_{i]m}) =$$
$$= T^{sm}_{..ki}q_{m}$$

when we find:

(25) $$2\tilde{b}^{sm}\,^{1}\nabla_{(k}b_{i)m} + \frac{2}{n-1}T^{se}_{..ik}\,^{1}\nabla_{[p}b_{e]m}\tilde{b}^{pm} = T^{sm}_{..ki}q_{m}$$

From (14) and (24) follows the equation

(26) $$q_{m} = \frac{2n}{n^{2}-1}\tilde{b}^{pe}(^{1}\nabla_{p}b_{em} - \frac{1}{n}\,^{1}\nabla_{m}b_{pe})$$

From the last two equations we have

$$\tilde{b}^{sm}\,^{1}\nabla_{(k}b_{i)m} = \frac{1}{n+1}\tilde{b}^{pe}T^{sm}_{..ki}\,^{1}\nabla_{(p}b_{m)e}$$

Since (10) is satisfied, we finaly obtain (22).

Theorem 4: If there is a GMC-correspondence between the coherences of spaces **^{1}An** and **^{2}An** and the two coherences are conjugated with respect to the tensor $\mathbf{b_{is}}$, then $\mathbf{b_{is}}$ satisfies the following condition

(27) $$^{1}\nabla_{[k}b_{i]s} = \frac{1}{n-1}\tilde{b}^{pm}(b_{ks}\,^{1}\nabla_{[p}b_{i]m} - b_{si}\,^{1}\nabla_{[p}b_{k]m})$$

Proof: Suppose the coherences of the two spaces are conjugated with respect to the tensor $\mathbf{b}_{is}$ and there is a GMC-correspondence between the spaces.

From (12) and (16) is follows that

$$\tilde{b}^{pm}(\delta_p^{s}\overset{1}{\nabla}_k b_{is} + \frac{2}{n-1}\delta_i^{s}\overset{1}{\nabla}_{[p} b_{k]m}) = \tilde{b}^{pm}(\delta_p^{s}\overset{1}{\nabla}_i b_{km} + \frac{2}{n-1}\delta_k^{s}\overset{1}{\nabla}_{[p} b_{i]m})$$

when we find

$$\tilde{b}^{sm}\overset{1}{\nabla}_{[k} b_{i]m} = \frac{1}{n-1}\tilde{b}^{pm}(\delta_k^{s}\overset{1}{\nabla}_{[p} b_{i]m} - \delta_i^{s}\overset{1}{\nabla}_{[p} b_{k]m})$$

After contracting with $\mathbf{b}_{is}$ in the above equation (27) is obtained.

REFERENCES

1. Норден А. П., Пространства афинной связности, изд. “Наука”, ГРФМЛ, М, 1976
2. Zlatanov, G. Z., Special networks in the n-dimensional Space of Weyl Wn, Tensor N. S. Vol. 48, No. 3, 1990, 241-247
3. G. Z. Zlatanov, B. Tzareva, Geometry of the Nets in Equiaffine Spaces, Journal of Geometry, Vol. 55, 1996, 192-201
4. Дубнов Я.С., Диагольные свойства сетей труды сем. по вект. и тенз. анализу, IX, 1952, 7-48
5. Дубнов Я.С. и Функс С. А., О пространственых аналогах чебышевской сети, ДАН, СССР, т. 28, No. 2, 102-104