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HYPERCOMPLEX VARIABLES

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The purpose of this note is to clarify some generalizations of the notion of complex number used in the papers [1] and [2]. We use ad hoc the term "hypercomplex number" having not a necessary erudition.

Let j be a symbol. We shall consider degrees of j, denoted j^k and subordinate only to the following formal conditions

$$j^0 = 1 \quad \text{and} \quad j^{2n} = -1.$$

By assumption we have an assotiative and commutative mutiplication for the degrees of j, i.e. $j^k j^l = j^{k+l}$, $k, l \in \mathbf{N}$. The sequence $j^0, j^1, j^2, \dots, j^{2n-1}$ is constitued by different elements.

1. THE ALGEBRA $\mathbf{R}[1, i, i^2, \dots, i^n]\,[1,j]$

We shall consider the vector space over the field of real numbers $\mathbf{R}$ consitituated by the following vectors

$$x = x_0 + x_1 j + x_2 j^2 + \dots + x_{2n-1} j^{2n-1},$$

where $x_k \in \mathbf{R}$, $k = 0, 1, 2, \dots, 2n-1$. There is a natural multiplicative operation in $\mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$ defined by the above introduced multiplication of the degrrees of j, namely

$$xy = (x_0 + x_1 j + \dots + x_{2n-1} j^{2n-1})(y_0 + y_1 j + \dots + y_{2n-1} j^{2n-1}) =$$

$$(x_0y_0 - x_1y_{2n-1} - \dots - x_{2n-1}y^1) + (x_0y_1 + x_1y_0 - \dots + \quad)j + \dots (x_{2n-1}y_0 + x_{2n-2}y_1 + \dots \quad)j^{2n-1}$$

So the vector space $\mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$ is a commutative and associative algebra with respect to mentioned multiplication.Having in mind that for each $x \in \mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$ we have

$$x = (x_0 + x_2 j^2 + \dots + x_{2n-2} j^{2n-2}) + (x_1 + x_3 j^2 + \dots + x_{2n-1} j^{2n-2})j$$

and setting $j^2 = i$ we obtain

$$x = (x_0 + x_2 i + \dots + x_{2n-2} i^{n-1}) + (x_1 + x_3 i + \dots + x_{2n-1} i^{n-1})j, \quad \text{or} \quad x = x' + x''j,$$

where $x' = x_0 + x_2 i + \dots + x_{2n-1} i^{n-1}$ and $x'' = x_1 + x_3 i + .. + x_{2n-1} i^{n-1}$.

The multiplication in $\mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$ induces a natural multiplication in $\mathbf{R}[1, i, \dots, i^{n-1}]$, namely the products $x'y'$ and $x''y''$. With respect to the induced multiplication, the obtained algebra is commutative, associative and distributive.

Having the algebra $\mathbf{A} = \mathbf{R}[1, i, i^2, \ldots, i^{n-1}]$, we obtain the following represantation of $\mathbf{R}[1, j, j^2, \ldots, j^{2n-1}]$ over $\mathbf{R}[1,i, i^2, \ldots, i^n]$

$$\mathbf{R}[1, j, j^2, \ldots, j^{2n-1}] = \mathbf{R}[1, i, i^2, \ldots , i^n] [1,j] ,$$

where $\mathbf{A}[1, j]$ is the vector space over $\mathbf{A}$ generated by 1 and j .

In the case $j^2 = -1$, i.e. n = 1, we have the isomorphisme $\mathbf{R}[1, j] = \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers. In this case the above remarked representation is banal, we can accept that $\mathbf{R}[1,i^0]$ is isomorphic to $\mathbf{R}$.

In terms of the mentioned representation, the operations in $\mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$ seem as follows

$$x + y := (x' + y') + (x'' + y'')j ,$$
$$xy := x'y' + x''y''i + (x'y'' + x''y')j, \; i = j^2, \; i^n = -1,$$

where $x = x' + x'j$ and $y = y' + y''j$.

The elements of $\mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$ will be called *hypercomplex numbers.* It is to remark that here i is not a complex number when $n \geq 2$.

2. CONJUGATION IN $\mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$

Let $x \in \mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$. By definition $\bar{x} := x' - x''j$. Clearly, we have

$$x \to x\bar{x}$$

PROPOSITION 1. The mapping $x \to \bar{x}$ is a linear and multiplicative involution of $\mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$.

Proof. It is easy o see that $\overline{x + y} = \bar{x} + \bar{y}$. Calculating $\bar{x}\,\bar{y}$ we get

$$\bar{x}\,\bar{y} = (x' + (-x'')j)(y' + (-y'')j) = x'y' + x''y''i - (x'y'' + x''y')j,$$

which coincides with the conjugate of the product xy, i.e. we have $\overline{xy} = \bar{x}\,\bar{y}$ □

CONSEQUENCE: $x\bar{x} = x'x' - x''x''i$.

It is to remark that $x'x'$ and $x''x''$ make sense in the algebra $\mathbf{A} = \mathbf{R}[1, i, \ldots, i^{n-1}]$.

In the case n = 1, we have $x = x_0 + x_1 j$, $\bar{x} = x_0 - x_1 j$, and $\bar{x}x = x_0^2 + x_1^2$. This is just the case of the complex numbers.

3. HYPERCOMPLEX AND PSEUDOMODULE STRUCTURES

The underllying vector space of the algebra $\mathbf{R}[1, j, j^2, \ldots, j^{2n-1}]$ is the real vector space $\mathbf{R}^{2n}$. We say that the algbra $\mathbf{R}[1, i, i^2, \ldots , i^n] [1,j]$ defines a *hypercomplex structure* on $\mathbf{R}^{2n}$ [3]. By definition, the $\mathbf{R}[1,i, \ldots , i^{n-1}]$-valued mapping

$$x \to x\bar{x}, \quad x \in \mathbf{R}[1, j, j^2, \ldots , j^{2n-1}],$$

is called a *pseudo-module structure* on $\mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$.

We set $D(x) := \bar{x}x$. Clearly, we have $D(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ is the origine of $\mathbf{R}^{2n}$.

The subset of $\mathbf{R}^{2n}$, defined by all x for which $D(x) = 0$, is called *zero set* of the pseudo-module structure D and will be denoted by H_D. Clearly, $\mathbf{0} \in H_D \subset \mathbf{R}^{2n}$.

In the case the zero set H_D reduces to the origine $\mathbf{0}$ the pseudo-module structure D is called a module structure.

PROPOSITION 2. The zero set H_D of the pseudo-module structure D coincides with the intersection of n quadratic surfaces in $\mathbf{R}^{2n}$.

Proof. As $D(x) = x'^2 + x''^2$, we have to calculate

$$(x_0 + x_2 i + x_{2n-2} i^{n-1})^2 + (x_1 + x_3 i + \dots + x_{2n-1} i^{n-1})^2$$

in $\mathbf{R}[1, i, \dots, i^{n-1}]$. The result of the mentioned calculation is of the form

$$P_0(x_0,\dots,x_{2n-1}) + P_1(x_0, \dots, x_{2n-1})i + \dots + P_{n-1}(x_0, \dots, x_{2n-1})i^{n-1},$$

where $P_k(x_0, \dots, x_{2n-1})$, $k = 0, 1, \dots, n-1$, are quadratic polynomials of the real variables $x_0, \dots, x_{2n-1}$. So $D(x) = 0$ is equivalent to the system

$$P_0(x_0,\dots,x_{2n-1}) = P_1(x_0, \dots, x_{2n-1}) = \dots = P_{n-1}(x_0, \dots, x_{2n-1}) = 0. \ \square$$

For instance, in the case $n = 3$ we have:

$$P_0(x_0,x_1,x_2,x_3,x_4,x_5) = x_0^2 + x_1^2 - 2(x_2x_4 + x_3x_5),$$
$$P_1(x_0,x_1,x_2,x_3,x_4,x_5) = x_1^2 + x_5^2 - 2(x_0x_2 + x_1x_2),$$
$$P_2(x_0,x_1,x_2,x_3,x_4,x_5) = x_2^2 + x_3^2 - 2(x_0x_4 + x_1x_5).$$

PROPOSITION 3. The pseudo-module structure D is a mutiplicative $\mathbf{R}[1,i, \dots, i^{n-1}]$-valued mapping, defined on $\mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$, which vanishes only on its characteristic set.

Proof. Let $x, y \in \mathbf{R}[1, j, j^2, \dots, j^{2n-1}]$, then

$$D(xy) = (xy)(\overline{xy}) = (xy)(\bar{x}\bar{y}) = (x\bar{x})(y\bar{y}) = D(x)D(y)$$

Let $x \notin H_D$, then $Dx) \neq \mathbf{0}$, because in the oposite case it follows that $x \in H_D$ $\square$

REMARK. For each n, $2 \leq n$, the mapping

$$x \to x\bar{x}$$

defines a pseudo-module which is not a module. Only for $n = 1$ the mentioned pseudo-module is a module, it is in fact $|z|^2$ with $z \in \mathbf{C}$.

EXAMPLES: 1.) $n = 2$, $\mathbf{R}[1, j, j^2, j^3]$, $j^4 = -1$, $\mathbf{A} = \mathbf{R}[1, i]$, $\mathbf{A}$ is isomorphic to $\mathbf{C}$. So $\mathbf{R}[1, j, j^2, j^3]$ is isomorphic to $\mathbf{C}[1, j]$.

This is the case considered in the papers [1] and [2].

2.) n=3, $\mathbf{R}[1, j, j^2, j^3, j^4, j^5]$, $j^6 = -1$, $\mathbf{A} = \mathbf{R}[1, i, i^2\}$, $\mathbf{A}$ is isomorphic to $\mathbf{C} \times \mathbf{R}$. Now the considered 6-dimensional algebra $\mathbf{R}[1, j, j^2, j^3, j^4, j^5]$ is isomorphic to $(\mathbf{C} \times \mathbf{R})[1, j]$.

The table of multiplication in **A** is the following

	1	*i*	*i²*
1	*1*	*i*	i^2
i	*i*	i^2	*-1*
i^2	i^2	*-1*	*i*

3.) n=4, $\mathbf{R}[1, j, j^2, j^3, j^4, j^5, j^6, j^7]$, $j^8 = -1$, $\mathbf{A} = \mathbf{R}[1, i, i^2, i^3]$, **A** is isomorphic to **CxC.** Here the considered 8-dimensional algebra is isomorphic to (**C**x**C**)[1, j].

4. MATRIX REPRESENTATION OF $\mathbf{R}[1,i, \ldots i^{n-1}][1, j]$

The considered algebra admits a natural matrix representation defined by the following mappings

$$j \to J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; j^2 \to J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \; j^{2n-1} \to J^{2n-1}.$$

We denote by *(x'), (x''), (x), (D(x))* the corresponding matrices, respectively, of the element *x', x'', x, D(x).*

$$x' \to (x') = x_0 + x_2 J^2 + x_4 J^4 + \cdots + x_{2n-2} J^{2n-2} = \begin{pmatrix} x_0 & 0 & x_2 & 0 & \cdots & x_{2n-2} & 0 \\ 0 & x_0 & 0 & x_2 & \cdots & 0 & x_{2n-2} \\ -x_{2n-2} & 0 & x_0 & 0 & \cdots & x_{2n-4} & 0 \\ 0 & -x_{2n-2} & 0 & x_0 & \cdots & 0 & x_{2n-4} \\ \vdots & & & & \ddots & \vdots & \vdots \\ -x_2 & 0 & -x_4 & 0 & \cdots & x_0 & 0 \\ 0 & -x_2 & 0 & -x_4 & \cdots & 0 & x_0 \end{pmatrix}$$

$$x'' \to (x'') = x_1 + x_3 J^3 + x_5 J^5 + \cdots + x_{2n-1} J^{2n-1} = \begin{pmatrix} x_1 & 0 & x_3 & 0 & \cdots & x_{2n-1} & 0 \\ 0 & x_1 & 0 & x_3 & \cdots & 0 & x_{2n-1} \\ -x_{2n-1} & 0 & x_1 & 0 & \cdots & x_{2n-3} & 0 \\ 0 & -x_{2n-1} & 0 & x_1 & \cdots & 0 & x_{2n-3} \\ \vdots & & & & \ddots & \vdots & \vdots \\ -x_3 & 0 & -x_5 & 0 & \cdots & x_1 & 0 \\ 0 & -x_3 & 0 & -x_5 & \cdots & 0 & x_1 \end{pmatrix}$$

The mentioned matrix representation $x \to (x)$ is a homomorphism of $\mathbf{R}[1,i, \ldots i^{n-1}][1, j]$ in the special algebra of all semi-circular matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{2n-1} \\ -a_{2n-1} & a_0 & a_1 & \cdots & a_{2n-2} \\ -a_{2n-2} & -a_{2n-1} & a_0 & \cdots & a_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & a_0 \end{pmatrix}$$

REMARK. In the case n = 1, we have the algebra **R**[1, j]. The above mentioned homomorphism gives just the well known matrix representation for complex numbers

$$z = x_0 + ix_1 \longrightarrow \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix}$$

The matrix (2 x 2) in the right side is an semiicircular matrix.

5. A REAL-NUMBER REPRESENTATION OF THE PSEUDO-MODULE STRUCTURE

According to the above developed homomorphism the pseudo-module D get the matrix representation

$$(D(x)) = (x'x' - x''x''J^2).$$

We set

$$\mu(x) = |\det(D(x))|.$$

The real non-negative numer $\mu(x)$ give a number-theoretic representation of the pseudo-module structure D(x). The non-negative function μ is multiplicative: $\mu(xy) = \mu(x)\mu(y)$.

HADAMAR'S TYPE THEOREM. The following inequalities hold

$$|det(x')|^2 \leq n(|x_0|^2 + |x_2|^2 + \ldots + |x_{2n-2}|^2),$$

$$|det(x'')|^2 \leq n(|x_1|^2 + |x_3|^2 + \ldots + |x_{2n-1}|^2),$$

$$\mu^2(x) \leq n \sum_{k=0}^{2n-1} |x_k|^2$$

This theorem is valide for general matrices, not only for semi-circular ones.
In the next we shall use a notion of norm for semi-circular matrices, namely

$$||(x)||^2 = \sum_{k=0}^{2n-1} |x_k|^2$$

So, we have $\mu^2(x) \leq n\,||(x)||^2$.

4. A COMPLEX-NUMBER REPRESENTATION FOR THE ALGEBRA **R**[1, i] [1,j].

The elements of **R**[1, i][1,j] are of the form x = $x_0 + x_2 i + (x_1 + x_3 i)j$, where i = j^2 In view that **R**[1, i] is isomorphic to the field of complex numbers **C**, the following complex-number presentation for the elements x ∈ **R**[1, i][1,j] holds

$$x \to z + wj, \text{ where } z := x_0 + x_2 i \text{ and } w := x_1 + x_3 i, \text{ or } x' \to z,\ x'' \to w.$$

Now the operations for hypercomplex numbers, i.e. the elements of **R**[1, i][1,j], are represented by its complex number compounents

$$x + y = z + u + (w + v)j, \quad xy = zu - wv + (zv + wu)j,$$

where y = u + vj , u,v ∈ **C**.

The conjugation in **R**[1, i][1,j] seems as follows $\bar{x} = z - wj$, and the corresponding pseudo-module structure D is represented by ordinary complex numbers: $x\bar{x} = z^2 - w^2 i$.

Clearly, the equality D(x) = 0 reduces to the equation

$$z^2 - w^2 i = 0,$$

which determines a complex surface in **CxC**. It is the zero set of the considered now hypercomplex structure. The corresponding pseudo-module μ(x) is determined by complex numbers

$$\mu^2(x) = |z^2 - w^2 i| , \quad x = z + wj, \; j^4 = -1.$$

In terms of real nuber matrix presentation we have

$$(x') = \begin{pmatrix} x_0 & 0 & x_2 & 0 \\ 0 & x_0 & 0 & x_2 \\ -x_2 & 0 & x_0 & 0 \\ 0 & -x_2 & 0 & x_0 \end{pmatrix}, \quad (x'') = \begin{pmatrix} x_1 & 0 & x_3 & 0 \\ 0 & x_1 & 0 & x_3 \\ -x_3 & 0 & x_1 & 0 \\ 0 & -x_3 & 0 & x_1 \end{pmatrix},$$

Calculating the corresponding to D(x) = x'x' - x"x"i matrix (D(x))=(x')(x')-(x")(x")J^2 we get

$$(D(x)) = \begin{pmatrix} x_0^2 - x_2^2 + 2x_1x_3 & 0 & x_1^2 - x_3^2 + 2x_0x_2 & 0 \\ 0 & x_0^2 - x_2^2 + 2x_1x_3 & 0 & x_1^2 - x_3^2 + 2x_0x_2 \\ -2x_0x_2 - (x_1^2 - x_3^2) & 0 & x_0^2 - x_2^2 + 2x_1x_3 & 0 \\ 0 & -2x_0x_2 - (x_1^2 - x_3^2) & 0 & x_0^2 - x_2^2 + 2x_1x_3 \end{pmatrix}$$

Then *(D(x))* = 0 is equivalent to the following system

$$x_0^2 - x_2^2 + 2x_1x_3 = 0 , \qquad x_1^2 - x_3^2 + 2x_0x_2 = 0$$

The solutions of this system determine the zero-set of the considered hypercomplex structure, now in terms of real quadratic polynomials.

5. POWER SERIES OF HAPERCOMPLEX VARIABLES

Let (x) be a semi-circular matrix, which will be considered as a *hypercomplex variable*. We shall concider formal power series of the following kind

$$S((x)) := \Sigma a_k(x)^k, \quad \text{where } a_k \text{ are real (complex) numbers, } k = 0, 1, 2, \ldots .$$

By definition S((x)) is convergent iff

$$\sum_k |a_k| (\mu(x))^k \leq +\infty$$

It is to recall that $\mu((x)^k) = (\mu(x))^k$.

EXAMPLE: The matrix power series $\sum_k 1/k!\ (x)^k$ is convergent for every fixed matrix (x).

Indeed, we have

$$\sum_k 1/k!\ \mu(x)^k = e^{\mu(x)}.$$

We set $$e^{(x)} := \sum 1/k!\ (x)^k .$$

Let f(x) be a real-valed or complex-valued function, defined on the set of hypercomplex numbers.

We say that f(x) is real-analytic (complex-analytic) function of hypercomplex variable if the corresponding function of the matrix (x), i.e. f((x)), admits a convergent power series matrix development S((x)).

We can remark that it is possible to develop the fundamental power series theory in the sketched above hypercomplex context.

7. CAUCHY-RIEMANN THEORY

According to [1] and [2], a mapping $f: \mathbf{R}^4 \to \mathbf{R}^4$ defines a pseudo-holomorphic function f on R^4 if the differential df commutes with J, $J^4 = -1$, i.e.

$$df \circ J = J \circ df.$$

The coordinate functions $f_k = f_k(x_0, x_1, x_2, x_3)$, k = 0,1,2,3, of f are satisfy a kind of Cauchy-Riemann equations, namely

$$\frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3}$$

$$\frac{\partial f_0}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_3} = -\frac{\partial f_3}{\partial x_0}$$

$$\frac{\partial f_0}{\partial x_2} = \frac{\partial f_1}{\partial x_3} = -\frac{\partial f_2}{\partial x_0} = -\frac{\partial f_3}{\partial x_1}$$

$$\frac{\partial f_0}{\partial x_3} = -\frac{\partial f_1}{\partial x_0} = -\frac{\partial f_2}{\partial x_1} = -\frac{\partial f_3}{\partial x_2}$$

Here we remark that the same definition make sense and in $\mathbf{R}^{2n}$.

Using an appropriate complexifications, it is shown in [2], that the real mapping described above can be represented as complex mappings of the form $\mathbf{C}^2 \to \mathbf{C}^2$. In this setting the Cauchy-Riemann conditions change appropriately.

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