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ON AN J-CONNECTION ON A B-MANIFOLD

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Let (M, g ,J) be a Riemannian manifold *M* with a metric *g* and a structure *J* such that J^2=-id, g(Jx, Jy)=-g(x, y), x, y $\in$ XM, $\nabla J=0$. Now we discuss another symmetric *J*-connection $\overline{\nabla}$, related with the connection ∇ of g. If $\overline{R}=0$, where $\overline{R}$ is the curvature tensor of $\overline{\nabla}$ we get a known subclass [1], [3] of the *B*-manifolds. The inverse problem is discussed too.

A Riemannian manifold M_{2n} is in the class *GB*, of the so called generalized *B*-manifolds [2], if M_{2n} admits an almost complex structure *J* and a *B*-metric *g*, i.e.

$$J^2=-id, \quad g(Jx, Jy)=-g(x, y),\ x, y \in XM.$$

If *M* is in *GB* and $\nabla J=0$, where ∇ is a Riemannian connection of *g*, then *M* is in the class *B* of the *B*-manifolds [4].

Let J_i^s and g_{is} be the local coordinates of *J* and *g* respectively. We note by $J_{ij}=J_i^s g_{sj}$ and by virtue of (1) we have $J_{is}=J_{si}$. So we can define another symmetric connection $\overline{\nabla}$, whose Ricci-Christoffel symbols are as follows:

$$\overline{\Gamma}_{ij}^{k}=\Gamma_{ij}^{k}+T_{ij}^{k}, \quad T_{ij}^{k}=g_{ij}f^k-J_{ij}\tilde{f}^{k}, \tag{2}$$

where f^k is a vector field, $\tilde{f}^k=J_t^k f^t$ and Γ_{ij}^k are the Ricci-Christoffel symbols of ∇. By a direct calculations we see, that $\overline{\nabla}$ is an *J*-connection (i.e. $\overline{\nabla} J=0$), but it is not metric-connection (i.e. $\overline{\nabla} g\neq 0$).

Now we consider the map $\alpha:\nabla\to\overline{\nabla}$ defined by (2). Let $\overline{R}$,R be the curvature tensor fields of $\overline{\nabla}$ and ∇ respectively. If *T* is the tensor of the affine deformation, then it is well known

$$\overline{R}^{h}_{ijk}=R^{h}_{ijk}+\nabla_j T^h_{ik}-\nabla_k T^h_{ij}+T^s_{ik}T^h_{sj}-T^s_{ij}T^h_{sk} \tag{3}$$

for the local coordinates of $\overline{R}$, R and *T* respectively.
From (2) and (3) it follows

$$\overline{R}^{h}_{ijk}=R^{h}_{ijk}+g_{ik}P^h_j-g_{ij}P^h_k-J_{ik}\tilde{P}^h_j+J_{ij}\tilde{P}^h_k, \tag{4}$$

where $P^h_j=\nabla f^h+f_j f^h-\tilde{f}_j\tilde{f}^h$, $\tilde{P}^h_j=P^t_j J_t^h$.

We note by $\overline{S}_{ij}=\overline{R}^p_{ijp}$, $S_{ij}=R^p_{ijp}$ the local coordinates of corresponding Ricci tensors $\overline{S}$ and S of $\overline{\nabla}$ and ∇. The functions $\overline{\tau}=\overline{S}_{ij}g^{ij}$, $\overline{\tau}^*=\overline{S}_{ij}J^{ij}$ and $\tau=S_{ij}g^{ij}$, $\tau^*=S_{ij}J^{ij}$ are the first and the second scalar curvatures of $\overline{\nabla}$ and ∇ respectively.

After a long calculation from (4) we find

$$2(1-n)\overline{S}_{ij} = 2(1-n)S_{ij} + 4(1-n)P_{ji} - (\overline{\tau} - \tau)\, g_{ij} + (\overline{\tau}^{*} - \tau^{*})\, J_{ij} \tag{5}$$

On the other hand we contract (4) by g^{ij} and using the above notations we set

$$\overline{S}_{ij} = S_{ij} + (2-2n)P_{ij}. \tag{6}$$

From (5) and (6) we get

$$4n(n-1)P_{ij} = (\tau - \overline{\tau})g_{ij} - (\overline{\tau}^{*} - \tau^{*})J_{ij} \tag{7}$$

Theorem 1. The tensor field Q defined as follows

$$Q^{h}_{ijk} = R^{h}_{ijk} + \frac{\tau}{4n(1-n)}(g_{ij}\delta_k^h - g_{ik}\delta_j^h + J_{ik}J_j^h - J_{ij}J_k^h) - \frac{\tau^{*}}{4n(1-n)}(g_{ij}J_k^h - g_{ik}J_j^h - J_{ik}\delta_j^h + J_{ij}\delta_k^h) \tag{8}$$

is invariant with respect to the map $\alpha:\nabla\to\overline{\nabla}$.
The proof follows from (4) and (7) immediately.

Now let $\overline{\nabla}$ be a locally flat connection. Then evidently $\overline{Q}=0$ and from Theorem 1 it follows $Q = 0$. So (8) implies

$$R(x, y, z, u) = \frac{\tau}{4n(n-1)}[g(x, u)g(y, z) - g(x, Ju)g(y, Jz) - g(y, u)g(x, z) + g(y, Ju)g(x, Jz)] + \frac{\tau^{*}}{4n(n-1)}[g(x, Jz)g(y, u) + g(y, Ju)g(x, z) - g(y, z)g(x, Ju) - g(y, Jz)g(x, u)]. \tag{9}$$

Thus we have the following assertion.

Theorem 2. Let M_{2n} be in B and $\alpha:\nabla\to\overline{\nabla}$ defined by (2). If $\overline{\nabla}$ is locally flat, then M_{2n} satisfies the identity (9).

Moreover from (9) we obtain $S_{ij} = \frac{\tau}{2n}g_{ij} - \frac{\tau^{*}}{2n}J_{ij}$, i.e. M_{2n} is an almost Einstein manifold.

The class of *B*-manifolds which satisfies (9) has been appeared at first in [1] and further in [3]. In [1] it has been proved that in this class the totally real section curvature is absolutely constant .

Now let have $\alpha:\nabla\to\overline{\nabla}$ defined by (2) but

(10) $$\nabla_i f_j = \frac{\tau}{4n(n-1)} g_{ij} - \frac{\tau^*}{4n(n-1)} J_{ij} - f_i f_j + \tilde{f}_i \tilde{f}_j ,$$

$$\nabla_i \tilde{f}_j = \frac{\tau}{4n(n-1)} J_{ij} + \frac{\tau^*}{4n(n-1)} g_{ij} - f_i \tilde{f}_j - \tilde{f}_i f_j .$$

Theorem 3. If ∇ satisfies (9) and f_n be a decision of (10), then $\overline{\nabla}$ is locally flat (n>2).

Proof. The integrability condition of (10) is satisfied identically by virtue of (9) . So (10) has at least one solution. Then from (4), (7), (9) and (10) there follows $\overline{R}$ =0, so the theorem is proved.

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