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CLASS OF MANIFOLDS ADMITTING HOLOMORPHICALLY–PROJECTIVE TRANSFORMATIONS

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We consider a class of B-manifolds M_{2n} ,which besides the metric g and the complex structure J admit a covector field f such that: $\nabla_i f_j = \alpha g_{ij} + \beta J_{ij}$, $\alpha,\beta \in FM_{2n}$. We state that such a manifold admits a non-trivial HP-transformation in some manifold of the same class.

We note by B the class of the complex Riemannian manifolds M_n such that

$$\overset{c}{\nabla}_{\lambda} \phi_{\mu} = A G_{\lambda\mu} \quad , \tag{1}$$

where $G_{\lambda\mu}$ is the metric of the manifold, A is non-zero function, ϕ_μ is a covector field on M_n such that $G^{\lambda\mu}\phi_\mu\phi_\lambda \neq 0$ and $\overset{c}{\nabla}$ is the connection of $G_{\lambda\mu}$. The above geometric objects are analytic with respect to some local coordinate system $\{z^\lambda = x^\lambda + ix^{\lambda+n}\}, \quad i^2 = -1, \quad \lambda \in \{1,2,...n\}$

Using the ideas and literally calculations from [1], where similar real differentiable objects are considered we can find the corresponding similar assertion, as follows:

Theorem 1. Let M_n be in B. Then M_n admits a non-trivial projective transformation with an analytic vector in some $\overline{M}_n$ in B.

Now we consider a class HB of real differentiable Riemannian manifolds M_{zn} . So M_{2n} is in HB, if M_{2n} is a B-manifold [2], i.e. M_{2n} admits a complex structure J_i^s, a B-metric g_{is} and J_i^s is covariantly constant with respect to the connection ∇ of g_{si} . That means:

$$J_i^s J_s^p = -\delta_i^p; \qquad J_i^p J_j^s g_{ps} = -g_{ij}; \qquad \nabla_k J_i^s = 0. \tag{2}$$

Moreover the manifold M_{2n} also satisfies the condition

$$\nabla_i f_j = \alpha g_{ij} + \beta J_{ij}, \tag{3}$$

where $J_{ij} = J_i^t g_{tj}$ and $\alpha,\beta \in FM_{2n}$, and f_i is a covector field on M_{2n} , such that $g^{is} f_i\ f_s \neq 0$; $g^{is} f_i \tilde{f}_s \neq 0$ here $\tilde{f}_s = J_s^i f_i$. From (3) taking account of (2) we get.

$$\nabla_i \tilde{f}_j = -\beta g_{ij} + \alpha J_{ij} \ . \tag{4}$$

All Latin indices run over the rang {1,2,.......2n}. On due to (2) we have $J_{is} = J_{si}$, consequently (3) and (4) shows there exist two functions f, $\tilde{f}$ in FM_{2n}, such that $f_i = \frac{\partial f}{\partial x^i}$, $\tilde{f}_i = \frac{\partial \tilde{f}}{\partial x^i}$, with respect to a local coordinate system $\{x^i\}$. By virtue of (2) the following is valid

(5) $$J_i^s = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}; \quad g_{is} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}; \quad J_{is} = \begin{pmatrix} -B & A \\ A & B \end{pmatrix},$$

where E is the unit matrix, A, B are symmetric matrices all of type (n,n).

Let R^s_{jki} and R_{is} be the curvature tensor and the Ricci tensor of ∇ respectively. We put $\alpha_i = \frac{\partial \alpha}{\partial x^i}$, $\beta_i = \frac{\partial \beta}{\partial x^i}$. From (3) we find

(6) $$f_s R^s_{jki} = \alpha_k g_{ij} - \alpha_i g_{kj} + \beta_k J_{ij} - \beta_i J_{kj} .$$

Contracting (6) by g^{ij} and $J^{ij}(= J^i_t g^{tj})$ respectively and using (2) we get

(7) $$-R^s_k f_s = (2n-1)\alpha_k - J^s_k \beta_s$$

(8) $$\tilde{R}^s_k f_s = -J^s_k \alpha_s - (2n-1)\beta_k .$$

From (7) and (8) it follows

(9) $$J^s_k \alpha_s = -\beta_k ; \qquad J^s_k \beta_s = \alpha_k .$$

Now we contract (6) with $f^j (= g^{js} f_s)$ and using (9) we obtain (10)

(10) $$\alpha_k f_i - \alpha_i f_k + \beta_k \tilde{f}_i - \beta_i \tilde{f}_k = 0 .$$

From (10) we get immediately

(11) $$\alpha_k \tilde{f}_i + \beta_i f_k - \beta_k f_i - \alpha_i \tilde{f}_k = 0$$

The equations (10), (11) show the following

(12) $$\alpha = \alpha(f, \tilde{f}); \qquad \beta = \beta(f, \tilde{f}); \qquad \frac{\partial \alpha}{\partial \tilde{f}} = \frac{\partial \beta}{\partial f}; \qquad \frac{\partial \beta}{\partial \tilde{f}} = -\frac{\partial \alpha}{\partial f} .$$

Let's accept the family $f(x^1, x^2, \ldots x^{2n}) = c_1$ for new coordinate hypersurfaces y^1=const and their orthogonal pats for coordinate lines y^1. So we have [1]

(13) $$f_i = \delta^1_i .$$

From (5) and (13) we get

(14) $$\tilde{f}_i = \delta^{n+1}_i .$$

That's why we accept $\tilde{f}(x^1, x^2, x^1, \ldots x^{2n}) = c_2$ for another new coordinate hypersurfaces and their orthogonal pats for coordinate lines y^{n+1}. Further the Greek indices will run over the rang {2,3,....n}. We put also $y^{\delta} = x^{\delta};\ y^{\delta+n} = x^{\delta+n}$. In the new coordinate system $\{y^i\}$ we have [1]

(15) $$g_{1\delta} = J_{n+1\delta} = 0; g_{n+1\delta} = J_{1\delta} = 0.$$

We note $\partial_i = \frac{\partial}{\partial y^i}$. Using (12) we conclude the following assertion is true .

Theorem 2. The complex function $A = \alpha - i\beta$ is analytic with respect to argument $z^1 = y^1 + iy^{n+1}$.

Further from (5) and (15) we find

(16) $$g^{11} = \frac{g_{11}}{g_{11}^2 + J_{11}^2} \quad ; \qquad J^{11} = \frac{J_{11}}{g_{11}^2 + J_{11}^2} \quad ;$$

$$g^{1\delta} = J^{1\delta} = g^{1n+1} = J^{1n+1} = 0.$$

We know the coefficients Γ of ∇ are as follows $\Gamma_{ij}^{s} = \frac{1}{2} g^{sp}(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij})$.

Using this formula, (2)-(4), (13)-(16) we get

(17) $$g^{1s}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}) = -2\alpha g_{ij} - 2\beta J_{ij}$$

$$g^{n+1s}\left(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}\right) = 2\beta g_{ij} - 2\alpha J_{ij} \ .$$

Putting in (17) i = 1, j = λ (or i = n+1; j = λ) and using (15) and (16) we get

(18) $$\partial_\lambda g_{11} = \partial_\lambda J_{11} = \partial_{\lambda+n} J_{11} = \partial_{\lambda+n} g_{11} = 0.$$

So $g_{11} = g_{11}\left(y^1, y^{n+1}\right)$ and $J_{11} = J_{11}(y^1, y^{n+1})$

Now putting in (17) i = j = 1 or i = 1, j = n+1; or j = n+1; i = λ respectively and comparing the obtained six equations we get

(19) $$\partial_1 g_{11} = \partial_{n+1} J_{11}; \partial_{n+1} g_{11} = -\partial_1 J_{11},$$

as well as after some computations we find

(20) $$\partial_1 \ln\left(g_{11}^2 + J_{11}^2\right) = -4\left(\alpha g_{11} + \beta J_{11}\right), \partial_1 \mathrm{arctg}\frac{g_{11}}{J_{11}} = -2\left(\beta g_{11} - \alpha J_{11}\right)$$

$$\partial_{n+1} \ln\left(g_{11}^2 + J_{11}^2\right) = -4\left(\beta g_{11} - \alpha J_{11}\right), \partial_{n+1} \mathrm{arctg}\frac{g_{11}}{J_{11}} == 2\left(\alpha g_{11} + \beta J_{11}\right).$$

If i = λ , j = μ+n , then (17) implies

(21) $$\partial_1 g_{\lambda\mu} = \partial_{n+1} J_{\lambda\mu}; \qquad \partial_{n+1} g_{\lambda\mu} = -\partial_1 J_{\lambda\mu}$$

as well as

(22) $$\partial_1 \ln\left(g_{\lambda_\mu}^{\ 2} + J_{\lambda_\mu}^{\ 2}\right) = 4\left(\alpha g_{11} + \beta J_{11}\right); \qquad \partial_1 \mathrm{arcg}\frac{g_{\lambda\mu}}{J_{\lambda\mu}} = 2\left(\beta g_{11} - \alpha J_{11}\right)$$

$$\partial_{n+1}\ln(g_{\lambda\mu}^2+J_{\lambda\mu}^2)=4(\beta g_{11}-\alpha J_{11})\,; \qquad \partial_{n+1}\mathrm{arctg}\frac{g_{\lambda\mu}}{J_{\lambda\mu}}=-2(\alpha g_{11}+\beta J_{11})\,;$$

From (18) and (19) there follows immediately

Theorem 3. The complex function $G_{11}(z^1)=g_{11}(y^1,y^{n+1})+i\,J_{11}(y^1,y^{n+1})$ is analytic.

Now comparing (20) and (22) after long computations we obtain

(23) $$g_{\lambda\mu}=e^{p_{\lambda\mu}}\left(\frac{J_{11}\sin g_{\lambda\mu}-g_{11}\cos q_{\lambda\mu}}{g_{11}^2+J_{11}^2}\right);\ J_{\lambda\mu}=e^{p_{\lambda\mu}}\left(\frac{J_{11}\cos q_{\lambda\mu}+g_{11}\sin q_{\lambda\mu}}{g_{11}^2+J_{11}^2}\right),$$

where $P_{\lambda\mu}$ and $q_{\lambda\mu}$ are arbitrary functions of $\left(y^{\delta},y^{\delta+n}\right)$ and we can choose them in the following way

(24) $$\partial_{\delta}p_{\lambda\mu}=-\partial_{\delta+n}q_{\lambda\mu}\ ; \qquad \partial_{\delta+n}p_{\lambda\mu}=\partial_{\delta}q_{\lambda\mu}\ .$$

Theorem 4. The complex functions $G_{\lambda\mu}=g_{\lambda\mu}+iJ_{\lambda\mu}$ are analytic with respect to $z^{\delta}=y^{\delta}+iy^{\delta+n}$, here λ,μ,δ are in {1,2,...,n}.

The proof following from (15), Theorem 3, (23) and (24).

The main purpose of the present paper is the following assertion.

Theorem 5. Let M_{2n} be in HB. Then M_{2n} admits a non-trivial holomorphically-projective transformation in some $\overline{M}_{2n}$ in HB.

Proof . Let M_{2n} be in HB. Then we consider the complex Riemannian manifold M_n^* with a metric $G_{\lambda\mu}$ from Theorem 4. It is easily to verify that M_n^* satisfies (1), where A is the function from Theorem 2 and $\phi_{\mu}=\delta_{\mu}^1+i\delta_{\mu}^{n+1}$. So M_n^* is in B. On due to Theorem 1 we have M_n^* admits a non-trivial projective transformation in some $\overline{M}_n^*$ in B. On the other hand following Norden [2] we obtain , that the real interpretation of M_n^* is M_{2n} and the real interpretation of $\overline{M}_n^*$ is some $\overline{M}_{2n}$ in HB. Finally the real interpretation of the projective transformation between M_n^* and $\overline{M}_n^*$ is a holomorphically-projective transformation between M_{2n} and $\overline{M}_{2n}$. So the theorem is proved.

REFERENCES

[1]. Синюков Н.С. Геодезические отображения римановых пространств. Москва, ,,Наука’’ 1979.

[2]. Норден А.П. Об одном классе четиримерных А пространств. Изв.вузов Матем. 1960, N4 .