CIRCUMSCRIBED AND ORTHOGONAL ARCHIMEDEAN CIRCLES IN GENERALIZED ARBELOS

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ABSTRACT

We will show that a generalized arbelos can be divided to n-aliquot parts by using orthogonal circles or circumscribed circles as same as by using inscribed circles. We will also show that there exist sextuplet circles of Archimedes in a generalized arbelos. We can move it continuously along the outer circle of a generalized arbelos by changing a coaxial system. These results provide many problems around circles. We introduce one of them in this paper.

1 PRELIMINARIES

Let Φ be a coaxial system and E be a point on the line passing thorough the centers of circles in Φ. We call the pair (Φ, E) a coaxial system with a fixed point. Let L and L' be the intersection points of Φ if it is the intersecting type, or the limiting points of Φ if it is the non-intersecting type and let f be a non-negative number with $f^2 = |EL| \cdot |EL'|$. Two points L and L' coincide if Φ is the tangent type. There exists a circle $\varepsilon \in \Phi$ with center E when Φ is the intersecting type, is the tangent type and $E \neq L$ or is the non-intersecting type and E is outside the closed interval between L and L'. The radius of the circle ε is f. We regard the circle ε as a point circle E when $E=L$ or $E=L'$, and as a virtual circle with radius fi when Φ is the non-intersecting type and E is between two points L and L', where i is the imaginary unit. We also denote by λ the radical axis of Φ and by $\boldsymbol{A}_\varepsilon$ the area outside the circle ε. If we take the line passing through the centers of circles in Φ as the x-axis and the radical axis as the y-axis, the coaxial system Φ is expressed as $\{x^2 - 2kx + y^2 + c = 0 | k \in \mathbf{R}\} \cup \{\text{the radical axis}\}$ for some real number $c \in \mathbf{R}$. We denote this coaxial system by Φ_c. It is the intersecting type, the tangent type or the non-intersecting type according as $c < 0$, $c = 0$ or $c > 0$. We have $f^2 = e^2 - c$ if $e^2 - c \geq 0$ and $f^2 = e^2 - c$ if $e^2 - c < 0$.

Let us take the above coordinate system and let e denote the x-coordinate of the point E. If the coaxial system Φ_c is the intersecting type, any circle $\alpha \in \Phi_c$ except ε has two intersection points with x-axis one of which is outside ε. We denote by $a(\alpha)$ the x-coordinate of this point. We define $a(\varepsilon) = e + \sqrt{e^2 - c}$ when $\alpha = \varepsilon$. If the coaxial system Φ_c is the non-intersecting type or the tangent type, any circle $\alpha \in \Phi_c$ has two intersection points with x-axis one of which is outside the closed interval $[-l, l]$. We denote by $a(\alpha)$ the x-coordinate of this point. For a member $\alpha \in \Phi_c$, we define the value $\mu(\alpha)$ as follows.

When $e^2 - c = 0$,

$$\mu(\alpha) = \begin{cases} \frac{1}{e - a(\alpha)} & if \quad \alpha \neq \lambda, \\ 0 & if \quad \alpha \neq \lambda. \end{cases}$$

When $e^2 - c \neq 0$,

$$\mu(\alpha) = \begin{cases} \frac{a(\alpha) - e - \sqrt{e^2 - c}}{a(\alpha) - e + \sqrt{e^2 - c}} & if \quad \alpha \neq \lambda, \\ 1 & if \quad \alpha \neq \lambda. \end{cases}$$

In the case $e^2 - c < 0$, the symbol $\sqrt{e^2 - c}$ means fi and the value $\mu(\alpha)$ is a complex number with $|\mu(\alpha)| = 1$. In this case we define $\mu_*(\alpha) = \frac{a' - e + fi}{a' - e - fi}$ for the member $\alpha \in \Phi_c$ and define $\omega(\alpha) = Arg(a' - e + fi)$ with $0 < \omega(\alpha) < \pi$, where a' is the x-coordinate of the intersection point of the member $\alpha \in \Phi$ and the x-axis different from the point $a(\alpha)$. Note that the above values depend on the choice of the point E but not on the choice of the coordinate system. For the members $\alpha, \beta \in \Phi_c$, we define the inequality as follows.

$$\alpha < \beta \quad \text{iff} \quad \begin{cases} \mu(\alpha) < \mu(\beta) & \text{if} \quad e^2 - c \geq 0 \\ \omega(\alpha) < \omega(\alpha) & \text{if} \quad e^2 - c < 0. \end{cases}$$

2 INSCRIBED CIRCLES, ORTHOGONAL CIRCLES AND CIRCUMSCRIBED CIRCLES

Let (Φ_c, E) be a coaxal system with a fixed point and γ be a circle with center E and radius g. For the members $\alpha, \beta \in \Phi_c$ we call a circle in $\boldsymbol{A}_\varepsilon$ touching both α and β a inscribed (*resp.* orthogonal or circumscribed) circle to γ with respect to α and β if it is touching and inside γ (*resp.* intersects γ orthogonally or touches γ externally). By the similar way in [5], [6] and [7] we have

Theorem 1 Assume that $\alpha < \beta$, $\alpha, \beta \not\subseteq \mathbf{R}^2 \setminus \boldsymbol{A}_\varepsilon$ and $\gamma \subseteq \boldsymbol{A}_\varepsilon$.

Then the radius of the inscribed circle to γ with respect to α and β is

$$\begin{cases} \dfrac{uv(\mu(\beta)-\mu(\alpha))}{u\mu(\beta)-v\mu(\alpha)} & \text{if } e^2 \neq c, \\ \dfrac{2u^2(\mu(\beta)-\mu(\alpha))}{1+2u(\mu(\beta)-\mu(\alpha))} & \text{if } e^2 = c. \end{cases}$$

The radius of the orthogonal circle to γ with respect to α and β is

$$\begin{cases} \dfrac{2uv(\mu(\beta)-\mu(\alpha))}{(u-v)(\mu(\beta)+\mu(\alpha))} & \text{if } e^2 \neq c, \\ 2u^2(\mu(\beta)-\mu(\alpha)) & \text{if } e^2 = c. \end{cases}$$

The radius of the circumscribed circle to γ with respect to α and β is

$$\begin{cases} \dfrac{uv(\mu(\beta)-\mu(\alpha))}{-v\mu(\beta)+u\mu(\alpha)} & \text{if } e^2 \neq c, \\ \dfrac{2u^2(\mu(\beta)-\mu(\alpha))}{1-2u(\mu(\beta)-\mu(\alpha))} & \text{if } e^2 = c. \end{cases}$$

Where $2u = g + \sqrt{e^2 - c}$ and $2v = g - \sqrt{e^2 - c}$.

Corollary1 Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be members of Φ_c with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and intersecting or touching the circle γ for all *j*. Then the following three conditions are equivalent.

1. Inscribed circles to γ with respect to α_{j-1} and α_j are congruent for all j.
2. Orthogonal circles to γ with respect to α_{j-1} and α_j are congruent for all j.
3. Circumscribed circles to γ with respect to α_{j-1} and α_j are congruent for all j.

Furthermore, these conditions are equivalent to that the sequence $\mu(\alpha_1), \mu(\alpha_2), \cdots, \mu(\alpha_n)$ is geometric if $e^2 \neq c$ or arithmetic if $e^2 = c$.

3 GENERALIZED ARBELOS IN N-ALIQUOT PARTS

Let α, β and γ be three circles such that the centers of these circles are collinear, α and β are inside γ and touching it at different points. We call such a configuration of three circles a generalized arbelos. Let Φ a coaxial system generated by the circles α and β and let E be the center of the circle γ. In this situation we have $\gamma \subseteq \boldsymbol{A}_\varepsilon$ since the points L and L' are inside the circle γ . As in the first section we can write $\Phi = \Phi_c$ for some real number c. Now we call a configuration of figures

$$\{\alpha = \alpha_0, \alpha_1, \cdot\cdot\cdot, \alpha_n = \beta, \gamma \mid \alpha_j \in \Phi_c, \alpha_0 < \alpha_1 < \cdot\cdot\cdot < \alpha_n\}$$

a generalized arbelos in n-aliquot parts if the conditions in Corollary1 hold. In the generalized arbelos in n-aliquot parts we call the inscribed circle to γ with respect to α_{j-1} and α_j $(1 \le j \le n)$ an Archimedean circle in n-aliquot parts. Also we call an orthogonal (resp. circumscribed) circle to γ with respect to α_{j-1} and α_j $(1 \le j \le n)$ an orthogonal (resp. circumscribed) Archimedean circles in n-aliquot parts. Then we have

Theorem 2 The radius of the Archimedean circle in n-aliquot parts is

$$\begin{cases} \dfrac{uv(u^{\frac{2}{n}} - v^{\frac{2}{n}})}{uu^{\frac{2}{n}} - vv^{\frac{2}{n}}} & \text{if } e^2 \neq c, \\ \dfrac{2u}{n+2} & \text{if } e^2 = c. \end{cases}$$

The radius of the orthogonal Archimedean circle in n-aliquot parts is

$$\begin{cases} \dfrac{2uv(u^{\frac{2}{n}} - v^{\frac{2}{n}})}{(u-v)(u^{\frac{2}{n}} + v^{\frac{2}{n}})} & \text{if } e^2 \neq c, \\ \dfrac{2u}{n} & \text{if } e^2 = c. \end{cases}$$

The radius of a circumscribed Archimedean circle in n-aliquot parts is

$$\begin{cases} \dfrac{uv(u^{\frac{2}{n}} - v^{\frac{2}{n}})}{-vu^{\frac{2}{n}} + uv^{\frac{2}{n}}} & \text{if } e^2 \neq c, \\ \dfrac{2u}{n-2} & \text{if } e^2 = c. \end{cases}$$

Theorem2 says that the radii of the above circles are determined by the radii of two circles γ and ε.

4 ARCHIMEDEAN CIRCLES

In a generalized arbelos $\{\alpha, \beta, \gamma\}$, an Archimedean circle in 2-aliquot parts has radius $uv/u+v = g^2 - f^2/4g$ which we denote by r_A. We call a circle having this radius an Archimedean circle. To find new Archimedean circles is an interesting problem and many authors have found many new Archimedean circles recently([1], [2], [4], [5], [6], [7], $\cdots$). In this section we make a continuous family of six Archimedean circles.

Let denote the tangent point of the circles γ and α (resp. β) by P_1(resp. P_{-1}) and let γ_1(resp. γ_{-1}) be a circle congruent to γ and touching it externally at the point P_1(resp. the point P_{-1}). Let P_2(resp. P_{-2}) be the center of γ_1(resp. γ_{-1}) and P_3(resp. P_{-3}) be the point with P_1P_3(resp. $P_{-1}P_{-3}$) being a diameter of the circle γ_1(resp. the circle γ_{-1}). Let λ' be a line perpendicular to the line passing through the points P_j and let Φ' be a coaxial system generated by the circle ε and the line λ'.

Theorem 3 Let α^o (resp. β^o) be a circle in the coaxial system Φ' with $\alpha^o < \lambda'$ (resp. $\lambda' < \beta^o$). Assume that the orthogonal circle to γ with respect to α^o and λ' (resp. λ' and β^o) exists. Then it is an Archimedean circle if and only if it passes through the points P_2(resp. the points P_{-2}).

Let α^c(resp. β^c) be a circle in the coaxial system Φ' with $\alpha^c < \lambda'$ (resp. $\lambda' < \beta^c$). Assume that the circumscribed circle to γ with respect to α^c and λ' (resp. λ' and β^c) exists. Then it is an Archimedean circle if and only if it passes through the point P_3 (resp. the point P_{-3}).

Now we start from the following configuration of circles.

Let γ_j ($j \in \boldsymbol{Z}$) be congruent circles of radius g such that γ_j and γ_{j+1} touches externally and the centers of them are collinear. Let denote the center of the circle γ_j by P_{2j} and the tangent point of the circles γ_j and γ_{j+1} by P_{2j+1}. Let ε_j be congruent circles with center $P_{2j}$and let f be their radii. We assume that $f < g$. In this section we take the coordinate system such that the points P_j are on the x-axis for all j and P_0 is the origin. Let $\Phi^{j,e}$ be a coaxial system generated by the circle ε_j and the line $\lambda(x = e)$ for $j \in \boldsymbol{Z}$ and $e \in \mathbf{R}$. Note that $\Phi^{j,e} = \Phi_{(2jg-e)^2-f^2}$.

Let $\alpha_{j,e}, \beta_{j,e}, \alpha^{o}_{j,e}, \beta^{o}_{j,e}, \alpha^{c}_{j,e}, \beta^{c}_{j,e}$ be members in the coaxial system $\Phi^{j,e}$ with $\alpha_{j,e} \ni P_{2j+1}, \beta_{j,e} \ni P_{2j-1}, \alpha^{o}_{j,e} \ni P_{2j+2}, \beta^{o}_{j,e} \ni P_{2j-2}, \alpha^{c}_{j,e} \ni P_{2j+3}, \beta^{c}_{j,e} \ni P_{2j-3}$, We denote by $C^{r}_{j,e}$(resp. $C^{l}_{j,e}$) the inscribed circle to γ_j with respect to $\alpha_{j,e}$ and λ(resp. λ and $\beta_{j,e}$), by $C^{r,o}_{j,e}$(resp. $C^{l,o}_{j,e}$) the orthogonal circle to γ_j with respect to $\alpha^{o}_{j,e}$ and λ(resp. λ and $\beta^{o}_{j,e}$), by $C^{r,c}_{j,e}$(resp. $C^{l,c}_{j,e}$) the circumscribed circle to γ_j with respect to $\alpha^{c}_{j,e}$ and λ(resp. λ and $\beta^{c}_{j,e}$).

There exists the circles $C^{r}_{j,e}$(resp. $C^{l}_{j,e}$) if and only if

$(2j-1)g \leq e \leq (2j+1)g - 2r_A$ (resp. $(2j-1)g + 2r_A \leq e \leq (2j+1)g$).

There exists the circles $C^{r,c}_{j,e}$(resp. $C^{l,c}_{j,e}$) if and only if

$(2j-1)g - 2r_A \leq e \leq (2j+1)g$ (resp. $(2j-1)g \leq e \leq (2j+1)g + 2r_A$).

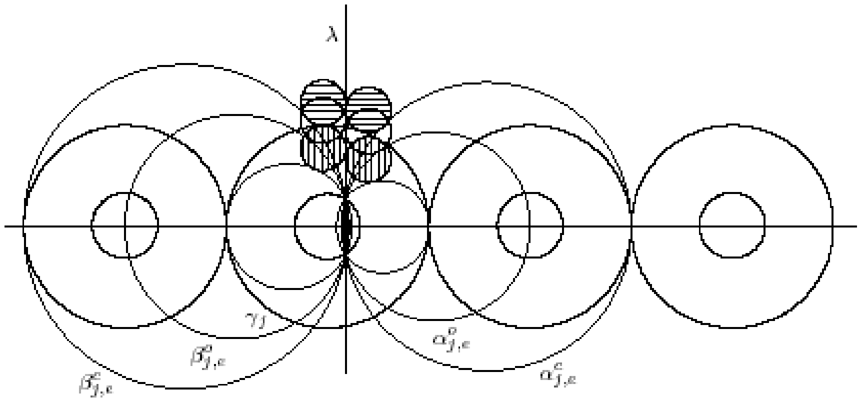
There exists the circles $C^{r,o}_{j,e}$(resp. $C^{l,o}_{j,e}$) if and only if

$2jg - r_A - \sqrt{r_A^2 + g^2} \leq e \leq 2jg - r_A + \sqrt{r_A^2 + g^2}$

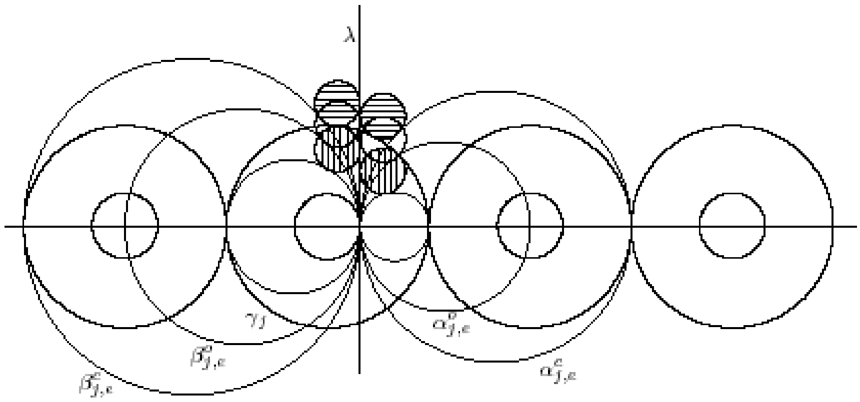
(resp. $2jg + r_A - \sqrt{r_A^2 + g^2} \leq e \leq 2jg + r_A + \sqrt{r_A^2 + g^2}$).

By Theorem3, the circles $C^{r}_{j,e}, C^{l}_{j,e}, C^{r,o}_{j,e}, C^{l,o}_{j,e}, C^{r,c}_{j,e}, C^{l,c}_{j,e}$ are all congruent having the radii r_A.

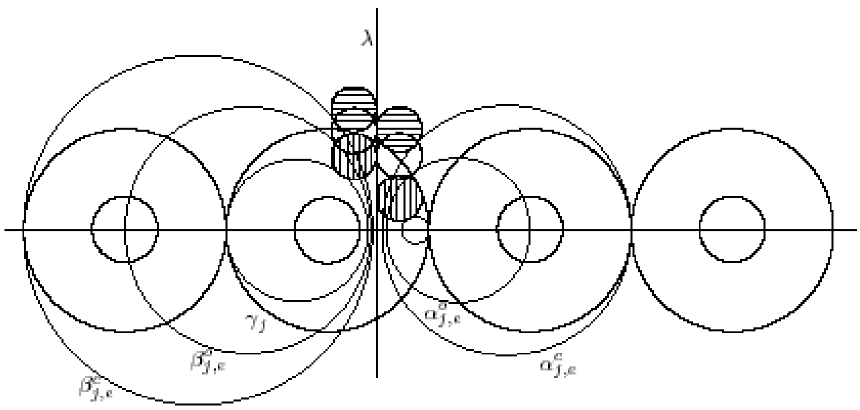
When $e = (2j+1)g - 2r_A$, the circle $\alpha^{c}_{j+1,e}$ passes through the point P_{2j+1} and the circle $\alpha_{j,e}$ degenerates to this point. So we have $C^{r}_{j,e} = C^{r,c}_{j+1,e}$. When $e = (2j+1)g + 2r_A$, the circle $\beta^{c}_{j,e}$ passes through the point P_{2j+1} and the circle $\beta_{j+1,e}$ degenerates to this point. So we have $C^{l}_{j+1,e} = C^{l,c}_{j,e}$.



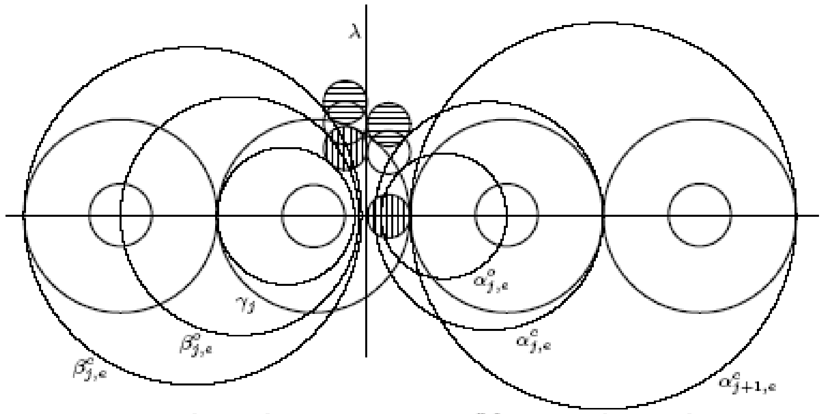
$2jg - f < e < 2jg + f$



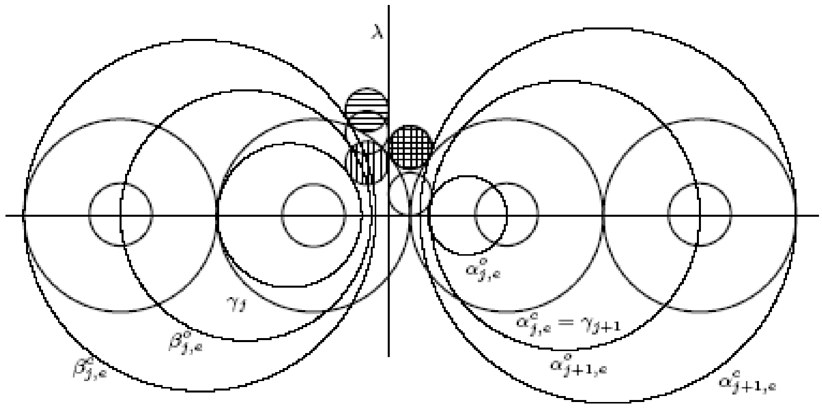
$e = 2jg + f$



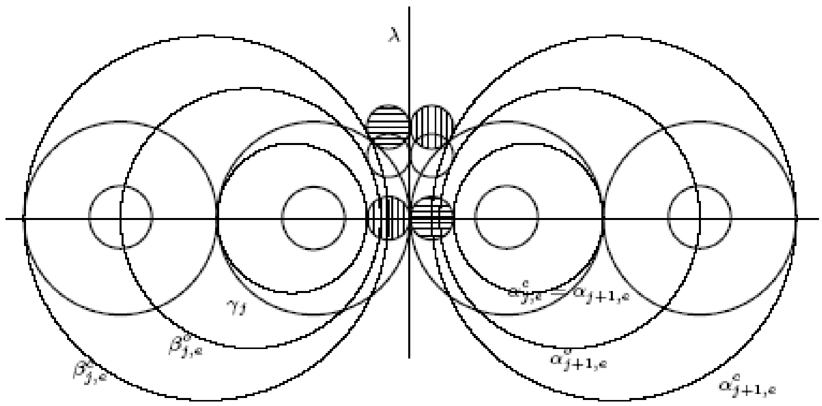
$2jg + f < e < (2j + 1) - 2r_A$



$e=(2j+1)g-2r_A\ ,\ \mathcal{C}^r_{j,e}=\mathcal{C}^{r,c}_{j+1,e}=\mathcal{C}^r_A\ (j\text{: even})$



$e=(2j+1)g-r_A\ ,\ \mathcal{C}^{r,o}_{j,e}=\mathcal{C}^{r,o}_{j+1,e}=\mathcal{C}^r_O\ ,\ \mathcal{C}^r_B=\mathcal{C}^{r,c}_{j,e}=\mathcal{C}^{r,c}_{j+1,e}=\mathcal{C}^r_A\ (j\text{: even})$



$e=(2j+1)g\ ,\ \mathcal{C}^r_{j+1,e}=\mathcal{C}^{r,c}_{j,e}=\mathcal{C}^r_B(j\text{: even})\ ,\ \mathcal{C}^l_{j,e}=\mathcal{C}^{l,c}_{j+1,e}=\mathcal{C}^l_A(j\text{: even})$

Figure 1

When $e = (2j+1)g$, we have $\Phi^{j,e} = \Phi^{j+1,e}$ since λ is the midline of the centers of ε_j and ε_{j+1}, so we have $\alpha_{j+1,e} = \alpha^c_{j,e}$ and $\beta_{j,e} = \beta^c_{j+1,e}$, and then $C^r_{j+1,e} = C^{r,c}_{j,e}$ and $C^l_{j,e} = C^{l,c}_{j+1,e}$. When $e = (2j+1)g - r_A$, both circles $C^{r,o}_{j,e}$ and $C^{r,o}_{j+1,e}$ have the same radii r_A and touches the *x*-axis at the point P_{2j+1}, so they are the same circle. Similarly, we have $C^{l,o}_{j,e} = C^{l,o}_{j+1,e}$ when $e = (2j+1)g + r_A$.

We define the circles $C^r_A, C^l_A, C^r_B, C^l_B, C^r_O, C^l_O$ of radii r_A as follows.

$$C^r_A = \begin{cases} C^r_{j,e} & \text{if } j \text{ is even and } (2j-1)g \le e \le (2j+1)g - 2r_A \\ C^{r,c}_{j,e} & \text{if } j \text{ is odd and } (2j-1)g - 2r_A \le e \le (2j+1)g \end{cases}$$

$$C^l_A = \begin{cases} C^l_{j,e} & \text{if } j \text{ is even and } (2j-1)g + 2r_A \le e \le (2j+1)g \\ C^{l,c}_{j,e} & \text{if } j \text{ is odd and } (2j-1)g \le e \le (2j+1)g + 2r_A \end{cases}$$

$$C^r_B = \begin{cases} C^{r,c}_{j,e} & \text{if } j \text{ is even and } (2j-1)g - 2r_A \le e \le (2j+1)g \\ C^r_{j,e} & \text{if } j \text{ is odd and } (2j-1)g \le e \le (2j+1)g - 2r_A \end{cases}$$

$$C^l_B = \begin{cases} C^{l,c}_{j,e} & \text{if } j \text{ is even and } (2j-1)g \le e \le (2j+1)g + 2r_A \\ C^l_{j,e} & \text{if } j \text{ is odd and } (2j-1)g + 2r_A \le e \le (2j+1)g \end{cases}$$

$$C^r_O = C^{r,o}_{j,e} \quad \text{if } (2j-1)g - r_A \le e \le (2j+1)g - r_A$$

$$C^l_O = C^{l,o}_{j,e} \quad \text{if } (2j-1)g + r_A \le e \le (2j+1)g + r_A .$$

These six circles are determined depending on the real number e and they move continuously according as e moves in **R**. The circles C^r_A and C^l_A move touching the circle γ_j internally and externally in turn and the circles C^r_B and C^l_B move touching the circle γ_j from the opposite side to C^r_A and C^l_A. The circles C^r_O and C^l_O move intersecting the circle γ_j orthogonally. We call these six circles sextuplet circles of Archimedes. Figure1 shows how these six circles move.

5 ARCHIMEDEAN CIRCLES IN EDUCATION

Theorem2 and Theorem3 lead many problems around circles for students. The author of this paper gave some lectures of plane geometry mainly about the theory of arbelos aiming for understanding ’coaxial systems’ and being able to deal with ’inversion’. In one of the lectures I gave students a following problem related to Theorem3.

Problem Let α, β, γ be circles with radii $a, b, a+b$ making the usual arbelos and let α', β' be two circles touching the circles α and β at their tangent point.

1. Find the condition that the circle touching both α' and β' and intersecting γ orthogonally is an Archimedean circle.

2. Find the condition that the circle touching both α' and β' and touching γ externally is an Archimedean circle.

The students in that class are freshmen and sophomore in the course of life science and informatics and the course of system life engineering in my university. Some of them dealt with an inversion to get a good results which says that the condition for the first case is $(3a + b/2p) - (a + 3b/2q) = 1$ and the condition for the second case is $(2a + b/p) - (a + 2b/q) = 1$, where p and q are the *x*-coordinates of the centers of $\alpha'$and β'([3]). These results contain a part of the results of the tangent case in Teorem3 and a part of the results of [4].

REFERENCES

[1] L. Bankoff., Are the twin circles of Archimedes really twins?, Math. Mag., 47, 214-218 (1974).

[2] C.W. Dodge, T. Schoch, P.Y. Woo, P.Yiu, Those ubiquitous Archimedean circles. Math. Mag., 72, 202-213 (1999).

[3] Y. Fukaya, Y. Yoshikawa, M. Watanabe, Orthogonal and circumscribed Archimedean circles in arbelos

[4] H. Okumura, M. Watanabe, The Archimedean Circles of Schoch and Woo, Forum Geom.,4, 27-34 (2004).

[5] H. Okumura and M. Watanabe, The Arbelos in *n*-Aliquot Parts, Forum Geom.,537-45 (2005).

[6] H. Okumura and M. Watanabe, Generalized Arbelos in Aliquot Parts: Intersecting case, J. Geometry Graphics 12, no.1, 53-62 (2008)

[7] H. Okumura and M. Watanabe, Generalized Arbelos in Aliquot Parts: Non-intersecting case, to appear in J. Geometry Graphics 13 (2009)