ON THE CONCEPT OF A QUADRILATERAL
IN THE SCHOOL GEOMETRY COURSE

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ABSTRACT

This article aims to show how a classification can be done concerning quadrilaterals in the school curriculum using systems of equalities and inequalities. It is also shown that it is easy to prove certain theorems through such a system. Our method provides solutions to a number of didactical problems [1] around the concept of a quadrilateral.

1. CRITERIA FOR EXISTENCE OF A QUADRILATERAL. GENERAL PROPERTIES

In this article we will consider only convex quadrilaterals.
Let us solve the following problem:
Construct a quadrilateral *ABCD* if the lengths of the four sides and a diagonal are given.

Without loss of generality, we may assume that BD is the given diagonal. The solution can be obtained through a construction of the triangles ABD and BCD. Each of them can be constructed by their given three sides. Let us denote:
$AB = a_1, BC = a_2, CD = a_3, DA = a_4$ and
$\sphericalangle DAB = \alpha, \sphericalangle BCD = \gamma, \sphericalangle ABD = \varphi_1, \sphericalangle CBD = \varphi_2, \sphericalangle CDB = \varphi_3, \sphericalangle ADB = \varphi_4$.
The four sides and the diagonal of the quadrilateral *ABCD* should satisfy the following system of conditions:

(1) $$\begin{cases} a_i > 0, i = 1, 2, ..., 5 \\ a_1 < a_4 + a_5, a_4 < a_1 + a_5 \\ a_5 < a_4 + a_1, a_2 < a_3 + a_5 \\ a_3 < a_2 + a_5, a_5 < a_2 + a_3 \end{cases}$$

These properties are drawn from the triangle inequality for both triangles. When we construct a special kind of triangles, we add to this system other equalities/inequalities. We involve further the cosine theorem and Heron's formula for the triangle area. For the triangles *ABD* and *BCD* we have:

(2) $$\cos\alpha = \frac{a_1^2 + a_4^2 - a_5^2}{2a_1a_4},$$

(3) $$\cos\varphi_1 = \frac{a_1^2 + a_5^2 - a_4^2}{2a_1a_5},$$

(4) $$\cos\varphi_4 = \frac{a_4^2 + a_5^2 - a_1^2}{2a_4a_5},$$

(5) $$S_{\vartriangle ABD} = \sqrt{p_1(p_1 - a_1)(p_1 - a_4)(p_1 - a_5)},$$

(6) $$\cos\gamma = \frac{a_2^2 + a_3^2 - a_5^2}{2a_2a_3},$$

(7) $$\cos\varphi_2 = \frac{a_2^2 + a_3^2 - a_5^2}{2a_2a_5},$$

(8) $$\cos\varphi_3 = \frac{a_3^2 + a_5^2 - a_2^2}{2a_3a_5},$$

(9) $$S_{\vartriangle BCD} = \sqrt{p_2(p_2 - a_2)(p_2 - a_3)(p_2 - a_5)},$$ where

(10) $$p_1 = \frac{a_1 + a_4 + a_5}{2},\ p_2 = \frac{a_2 + a_3 + a_5}{2},\ S_{ABCD} = S_{ABD} + S_{BCD}.$$

2. CONDITIONS FOR SOME KINDS OF QUADRILATERALS

1. Inscribed quadrilateral:
   Since $\alpha + \gamma = \pi \Rightarrow \cos\alpha = -\cos\gamma$. Substituting in this from (2) and (6), and after some standard transformations, we obtain

(11) $$(a_1a_2 + a_3a_4)(a_1a_3 + a_1a_4) = a_5^2(a_1a_4 + a_2a_3).$$

2. Circumscribed quadrilateral:

(12) $a_1 + a_3 = a_2 + a_4.$

3. Quadrilateral with perpendicular diagonals:

(13) $a_1^2 + a_3^2 = a_2^2 + a_4^2.$

4. Trapezium:

Since $a_1 \parallel a_3 \Rightarrow \cos\varphi_1 = \cos\varphi_3.$ Substituting (3) and (8) in this, we obtain

(14) $a_3\left(a_1^2 + a_5^2 - a_4^2\right) = a_1\left(a_3^2 + a_5^2 - a_2^2\right).$

5. Similarly, $a_2 \parallel a_4$ leads to

(15) $a_4\left(a_2^2 + a_5^2 - a_3^2\right) = a_2\left(a_4^2 + a_5^2 - a_1^2\right).$

6. At least one angle of the quadrilateral is a right one:

From $\alpha = 90^\circ$ *and* (2) $\Rightarrow$

(16) $a_1^2 + a_4^2 = a_5^2.$

From $\gamma = 90^\circ$ *and* (6) $\Rightarrow$

(17) $a_2^2 + a_3^2 = a_5^2.$

From $\varphi_1 + \varphi_2 = 90^\circ$ *and* $\varphi_3 + \varphi_4 = 90^\circ \Rightarrow$

$\cos\varphi_1 = \sin\varphi_2$ and $\cos\varphi_3 = \sin\varphi_4.$

Using the last equalities, the identity $\sin\varphi = \sqrt{1-\cos^2\varphi}$, statements (3) and (7), (4) and (8), we obtain:

(18) $a_2\left(a_1^2 + a_5^2 - a_4^2\right) = a_1\sqrt{4a_2^2a_5^2 - \left(a_2^2 + a_5^2 - a_3^2\right)^2}$ and

(19) $a_4\left(a_3^2 + a_5^2 - a_2^2\right) = a_3\sqrt{4a_4^2a_5^2 - \left(a_4^2 + a_5^2 - a_1^2\right)^2}.$

7. Quadrilateral with equal sides:

(20) $a_1 = a_3,$

(21) $a_2 = a_4,$

(22) $a_1 = a_2,$

(23) $a_1 = a_4,$

(24) $a_3 = a_4,$

(25) $a_2 = a_3.$

We can teach different kinds of quadrilaterals to students, using the respective conditions among (11) – (25). They simply must be added to the system (1). If we add *k* conditions, where $1 \le k \le 15$, we will derive different systems, whose number is the number of combinations of 15 elements, class *k*, which equals to 32,768. Some of the systems will be equivalent.

a) quadrilateral – (1);
b) inscribed quadrilateral – (1), (11);
c) trapezium – (1), (14) or (1), (15);
d) parallelogram – (1), (14), (15);
e) rectangle – (1), (14), (15) and (16), or (1), (14), (15) and (17), or (1), (14), (15) and (18), or (1), (14), (15) and (19).

3. THEOREM PROOFS

1. Theorems of Ptolemy for inscribed quadrilaterals.
Set $AC = a_6$.

Theorem 1: $a_1a_3 + a_2a_4 - a_5a_6 = 0$.

Theorem 2: $\frac{a_5}{a_6} = \frac{a_1a_4 + a_2a_3}{a_1a_2 + a_3a_4}$.

Proof: Using (11) we derive for a_6 the following:

(26) $a_6\left(a_1a_2 + a_3a_4\right) = \left(a_1a_4 + a_2a_3\right)\left(a_1a_3 + a_2a_4\right)$.

We divide (26) to (11) and after standard transformations we obtain $\frac{a_5^2}{a_6^2} = \frac{\left(a_1a_4 + a_2a_3\right)^2}{\left(a_1a_2 + a_3a_4\right)^2}$, from which we get $\frac{a_5}{a_6} = \frac{a_1a_4 + a_2a_3}{a_1a_2 + a_3a_4}$.

Proof of Theorem 1: Using (26) we obtain

$$a_1a_3 + a_2a_4 - a_5a_6 = \frac{\left(a_1a_3 + a_2a_4\right)\left(a_1a_2 + a_3a_4\right) - \left(a_1a_4 + a_2a_3\right)a_5^2}{a_1a_2 + a_3a_4}.$$

The numerator of the fraction is zero, having in mind (11).

2. Finding conditions when a quadrilateral with perpendicular diagonals is circumscribed.

Let *O* be the point of intersection of the diagonals *AC* and *BD*. Let $AC = a_6$, $AO = k_1a_6$, $BO = k_2a_6$. Obviously, $0 < k_{1,2} < 1$. Then we have

$$a_1 = \sqrt{k_1^2a_6^2 + k_2^2a_5^2}, \quad a_2 = \sqrt{\left(1-k_1\right)^2 a_6^2 + k_2^2a_5^2},$$

$$a_3 = \sqrt{\left(1-k_1\right)^2 a_6^2 + \left(1-k_2\right)^2 a_5^2}, \quad a_4 = \sqrt{k_1^2a_6^2 + \left(1-k_2\right)^2 a_5^2}.$$

All these substituted in (12), give

$$\sqrt{k_1^2a_6^2+k_2^2a_5^2}+\sqrt{\left(1-k_1\right)^2a_6^2+\left(1-k_2\right)^2a_5^2}=\sqrt{\left(1-k_1\right)^2a_6^2+k_2^2a_5^2}+\sqrt{k_1^2a_6^2+\left(1-k_1\right)^2a_5^2}.$$

After rising to second power twice, we obtain

$$\left[k_1^2-\left(1-k_2\right)^2\right]\left[\left(1-k_2\right)^2-k_2^2\right]a_5^2a_6^2=0.$$

Hence $k_1^2-\left(1-k_2\right)^2=0$ or $\left(1-k_2\right)^2-k_2^2=0.$

Therefore $k_1=\frac{1}{2}$ or $k_2=\frac{1}{2}$, i.e. only deltoids, rhombs and squares can be with perpendicular diagonals and circumscribed. Deltoids are described through (1), (13), (20) and (21); or through (1), (13), (22) and (23).

3. Trapezium with perpendicular diagonals cannot be circumscribed.
Repeating the reasoning from above and having in mind that $k_1=k_2=k$, we obtain the identity $\left[\left(1-k\right)^2-k^2\right]a_5^2a_6^2=0$, which holds true for $k=\frac{1}{2}$. This is impossible for any trapezium.

4. The opposite sides of a parallelogram are equal.
The proof can be done by using (14), (15).

4. FINAL NOTES

This classification method for quadrilaterals, studied in school, makes the teacher's job easier, when he or she trains gifted students for exams and competitions [2]. The conditions expressed by (11) - (21) are also sufficient for the existence of the respective type of quadrilaterals. Thus, the teacher is able to introduce to the students a new class of quadrilaterals adding to (1) the respective conditions. If the new system has a solution, then the respective class of quadrilaterals does not exist. The teacher is also able to investigate new properties of the quadrilaterals examined in this article. The described method provides opportunities for algorithmic and computer added classification and investigation of quadrilateral type.

REFERENCES

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