MODELS OF THE REAL PROJECTIVE PLANE AND LINEAR COMBINATION OF THEM

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ABSTRACT

At first we revisit some well-known models of the real projective plane (RP_2). Then we introduce the S_1-and S_2-surfaces. The first one is a model of the real projective plane. The second one is a sphere. Using the well known Steiner's Roman surface and the Crosscap surface, we define linear combinations of these four surfaces and find some flat surfaces, especially two cones. Taking again a linear combination of these cones, we prove that such an arbitrary linear combination is again a cone.
For our investigations we apply the powerful computer algebra system MAPLE for calculations, visualizations and animations.

I. STEINER'S ROMAN SURFACE

The Roman map is:

(1) $$y_1 = x_2x_3, y_2 = x_3x_1, y_3 = x_1x_2.$$

The Jacobian of this map is

(2) $$Jy = 2x_1x_2x_3.$$

Steiner's Roman surface is the restriction of this map on the half sphere. On takes the half sphere because the Roman map is an antipodal map.
For the half sphere we take (as in the next models):

(3) $x_1 = \sin v \cos u, x_2 = \sin v \sin u, x_3 = \cos v$,

($v \in [-\frac{\pi}{2}, \frac{\pi}{2}], u \in [0, \pi]$).

We give some facts of this surface. It is well known that this surface is a model of the real projective plane (RP_2) and the origin ($(0,0,0)$ is its triple point. We give here some additional information about the Roman map. The half maximal circles $u = const$ are mapped on Steiner's Roman surface as an ellipsis through the origin. Solving the equation

(4) $$ax_1 + bx_2 + cx_3 = 0$$

with respect to v, we get

(5) $$V_{abc} = -\arctan(\frac{c}{a\cos u + b\sin u}).$$

It follows that the plane (4) cuts the half sphere along the half meridian $m(v = V_{abc})$. The corresponding curve, called **straight line** on the Steiner's Roman surface, is an ellipse. So all straight lines on this surface are ellipsis, some of them are going through the origin. Every two ellipsis (straight lines) have exactly one common point. We give a program for animation of the 1-parametrical set ($b = 2, c = 1$) of the above family.

II. CROSCAP SURFACE

The crosscap map is

(6) $$z_1 = x_2x_3, z_2 = 2x_1x_2, z_3 = x_1^2 - x_2^2.$$

The Jacobian of this map is

(7) $$Jz = -4(x_1^2 + x_2^2)x_2.$$

The **crosscap surface** is the restriction of this map on the half sphere.

The crosscap surface is also a model of the RP_2 , whose straight lines have the same properties as in the previews case.

These surfaces are threaded in [1]. From this paper, we use the following statement:

"To realize the real projective plane as a surface in the Euclidean space R^3, one can take a map $F : \sigma \to R^3$ of the unit sphere which has the antipodal property. Of course, we should choose F so that its Jacobian matrix has a zero determinant at only a few points. This can be accomplished by choosing the components of F to be certain quadratic polynomials". In this way one provides the bijectivity of the map.

III. THE SURFACE S_1

S_1 -map is the map

(8) $$u_1 = x_1(x_2 + x_3), u_2 = x_2(x_3 + x_1), u_3 = x_3(x_1 + x_2).$$

The Jacobian of this map is

(9) $$\det(S_1) = 4x_1x_2x_3.$$

S_1 **-surface** is the restriction of this map on the half sphere. It is defined by the first author. On the sphere we have

(10) $$\det(S_1) = 4\sin u \cos u \cos v \sin^2 v.$$

It is zero in the following cases:

a/ $v = \dfrac{\pi}{2}$; b/ $v = 0$; c/ $u = 0$, d/ $u = \dfrac{\pi}{2}$.

In accordance with these cases on the sphere we have:

a/ $(\cos u, \sin u, 0)$, b/ $(0,0,1)$, c/ $(\sin v, 0, \cos v)$, d/ $(0, \sin v, \cos v)$

and on the surface S_1 **:**

a/ $(\sin u \cos u, \sin u \cos u, 0)$, b/ $(0,0,0)$,
c/ $(\sin v \cos v, 0, \sin v \cos v)$, d/ $(0, \sin v \cos v, \sin v \cos v)$.

The mapping between the corresponding images on the sphere and on the surface S_1 is bijective, so the S_1 map is anywhere bijective between the sphere and the surface S_1 except the origin which is its triple point. To investigate this surface in detail, we use the MAPLE computer software and prove:

THEOREM 1. The S_1 -surface is non-orientable and it is a model of the RP_2.

This surface is visualized by Fig.1.

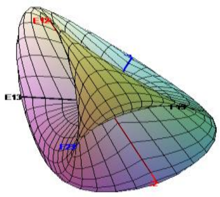

Fig 1

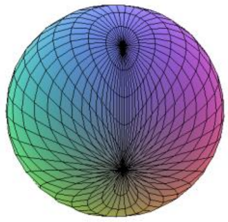

Fig.2

IV. THE SURFACE S_2

S_2 -map is the map

(11) $$v_1 = x_1(x_1 + x_2 + x_3), v_2 = x_2(x_1 + x_2 + x_3), v_3 = x_3(x_1 + x_2 + x_3)$$

S_2 **-surface** is the restriction of this map on the half sphere. It is also defined by the first author. It is a sphere and is shown by Fig. 2.

V. LINEAR COMBINATION OF THESE SURFACES

If a,b,c,d are arbitrary real numbers, we introduce the linear combination $S(a,b,c,d)$ of surfaces:

(12) $$w_i = ay_i + bz_i + cu_i + dv_i \, , i = 1,2,3,4 \, .$$

Using the Maple computer software, we find the Gaussian curvature $K(a,b,c,d)$. We prove:

(13) $$K(0,0,0) = \frac{4}{3} \, .$$

It shows S_2 is a sphere with radius

(14) $$r = \frac{\sqrt{3}}{2} \, .$$

We omit to write the corresponding expression for $K(a,b,c,d)$ because it is very long. But we prove the following

THEOREM 2. In the following cases

$$H: a=3, b=0, c=0, d=1; J: a=0, b=0, c=3, d=-1, \tag{15}$$

the surfaces $S(a,b,c,d)$ are flat and, more precisely, they are cones.
For the corresponding maps we have:

$$\begin{aligned} H_1 &= 3x_2x_3 + x_1(x_1+x_2+x_3), \\ H_2 &= 3x_3x_1 + x_2(x_1+x_2+x_3), \\ H_3 &= 3x_1x_2 + x_3(x_1+x_2+x_3) \end{aligned} \tag{16}$$

$$\begin{aligned} J_1 &= 3x_1(x_2+x_3) - x_1(x_1+x_2+x_3), \\ J_2 &= 3x_2(x_3+x_1) - x_2(x_1+x_2+x_3), \\ J_3 &= 3x_3(x_1+x_2) - x_3(x_1+x_2+x_3) \end{aligned} \tag{17}$$

Under the restriction on the unit half sphere, we get the corresponding cones represented by Fig 3 and Fig.4:

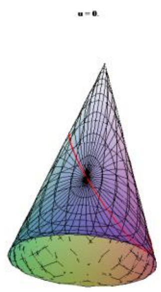

Fig. 3

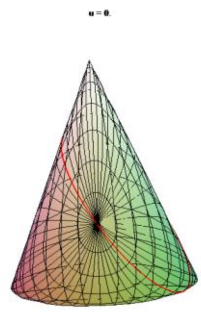

Fig. 4

We consider again some linear combinations -namely those of both cones $T(p,q)$:

$$\begin{aligned} T_1 &= pH_1 + qJ_1, \\ T_2 &= pH_2 + qJ_2, \\ T_3 &= pH_3 + qJ_3. \end{aligned} \tag{18}$$

Now we can prove the following unexpected

THEOREM 3. Any surface $T(p,q)$ is flat; more precisely: if $p=q$, the surface is a segment, and if $p \neq q$, the surface is again a cone. In a nutshell: a linear combination of cones is cone.

Proof. At first we calculate that the Gaussian curvature of the arbitrary surface (18) is zero. So the surface is flat. Then we investigate the behaviour of the normal vector field of (18). We show it is not constant and it has points which are not regular.

Fig. 5 illustrates the case $p=10, q=1$.

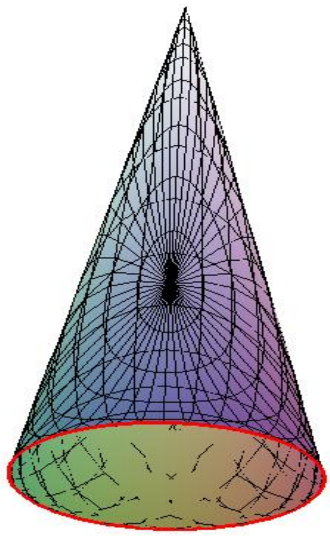

Fig. 5

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3. G. Stanilov, Sl. Slavova: Three new models of the real projective plane, International Science-Educational Conference: Mathematics, Physics, Informatics, Moscow, 23-27.03.2009.