ON TWO FUNDAMENTAL APPROACHES TO THE DEVELOPMENT OF SCIENTIFIC KNOWLEDGE AND THEIR IMPLEMENTATION IN DIDACTICS OF MATHEMATICS

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ABSTRACT

This study considers the appearance and the progress of two fundamental approaches to the development of scientific knowledge. According to the authors, each of them represents a complex formation of three simpler approaches which turn out to become requirements later. Regarding the terminology with respect to the functions of these formations in scientific and cognitive activities, the authors use the following names for the components of both approaches: "by-elementally", "repeat", "dependence" and "understanding", "convicting", "economizing" respectively. The two approaches themselves will be named "triads". The present study arguments the thesis that the second triad gives birth of three requirements which have been transformed into norms in structuring and representing mathematical knowledge since Ancient Greek times. The requirements are the following: to define notions by using primary or already defined notions only; to prove assertions by using axioms or already proved assertions only; to use already proved assertions as theorems without proving them each time. This study arguments also the thesis that the two triads are crucial in Mathematics Education for the formation of proving and heuristic style of thinking. In addition, during the last decades, the two fundamental approaches-triads have been used more and more consciously in the scientific and cognitive activities related to Didactics of Mathematics.

1. AN IDEA FROM THE DEVELOPMENT OF GEOMETRY IN THE PRE-GREEK AND THE FIRST CENTURIES OF THE ANCIENT GREEK PERIOD

Very often a general approach on different levels is applied to the development of scientific knowledge. This can be noticed apparently in the history of geometry development of the pre-Greek and the first centuries of the Ancient Greek period. Firstly, "entire" objects were studied and properties of separate elements of them were discovered. Based on such properties, a resemblance was noticed among some of the objects. This was in fact the "repeat" by type (form). A common name was attributed to a given type: "lying field", "direct field", "bull lob", "pyramid" (in fact this words comes from Ancient Egypt), "basket", etc. Thus, two approaches were initialized, viz. the "elemental" and the "repeat" approaches. Abstractions of basic geometric notions from directly used real objects began to be done by it. Also, it was noticed that some of the properties of entire objects are connected. For example, if the opposite sides of a "lying field" do not come closer or do not go far from each other, either, then their lengths are equal. If in addition the opposite vertices can be connected by equal sticks, then a "direct field" is obtained. Investigating and using similar connections, people practically added one more approach in addition to the "elemental" and "repeat" ones, namely the "dependence" approach. All this forms the "elemental, repeat and dependence" triad. Afterwards, when separate elements of entire objects were studied, the cycle was repeated, usually on a new level. Including the language, people directed themselves to the following two basic activities:

a) description of entire figures as composed of some elements and this lead to phrases, which gave birth to definitions;

b) formulation of assertions for element connections in entire figures or assertions for different figure connections, proves of the assertions and this lead to phrases, which gave birth to theorems.

The general approach under consideration and the described basic activities could be noticed in the development of the Language science, in Chemistry, Anatomy and others. Of course, it first happened in the Language science in that it paid attention to separate words, thus resulting in the appearance of hieroglyph writing. Later, it was noticed that words were composed of separate sounds, which were repeated in different combinations regarding different words. In Chemistry it was noticed, respectively, that various substances consisted of repeated elements, the number of which had been increased step by step, thus becoming more than 100.

It is not difficult to conclude that the "elemental, repeat and dependence" approach - triad assures an easier understanding of objects because in fact the cognitive methods of "analysis", "abstraction" and "synthesis" are applied systematically by it. What happens in analysis is the division of an object into parts (elements), then what comes is the attentive concentration on one or more separate parts disregarding the others and finally it is the turning back to the initial object. Thus, the object becomes known better for individual applications or for considerations as parts of more complex objects. The approach is effective not only in the investigation of new objects but also in teaching (study by help). In the second case what comes to the stage is the necessity of assuring "understanding of what is in course of explaining and convicting in its truthiness".

2. THE LANGUAGE AS A TOOL IN COGNITIVE ACTIVITY

In cognitive activity – individual or by teaching – mankind has created and has used another powerful device, namely the language. The role of the language is to fix knowledge from one part and to transfer knowledge from the other one. We are interested in the second part since we deal with Education. It turns out in it that two basic approaches have taken advantage:

a) pointing out of the object under observation and announcing its name that has been accepted by previous generations;

b) verbal description of the object under observation and announcing its name that has been accepted by previous generations.

The first approach is known to be an ostensive one and has been used since the most ancient times.

Everyday life practice has taught people that a successful activity in the second approach exists only if previously well known words are used in the description. In such a case, a psychic condition could be reached which the person under teaching would evaluate by the phrase "I have understood". The phenomenon has been repeated a million times when children are taught in their mother language by experienced people. Recently it was examined and described in [14] by the Russian scientist N. Kotova, who teaches Bulgarian language in the Moscow State University. Prof. Kotova carried out the following experiment with inhabitants of a village near Sofia. She chose special words and asked the villagers to explain them. Two groups of words were chosen. The first one contained the names of objects or activities that were available for the moment. At all the times the villagers pointed out the corresponding object or activity. For example, the reaction to the question "what is a canine tooth" was an immediate opening of the mouth and showing a canine tooth in it. The second group contained the names of objects or activities that were not available for the moment. The reaction in such cases was a description by words which were supposed to be known to the asking person. Analogously, villagers reacted to questions connected with abstract objects. The conclusions of this experiment are very important for Didactics.

3. THE TRIAD "UNDERSTANDING, CONVICTING AND ECONOMIZING" AS A PRINCIPAL ACTION FORMED IN ANCIENT GFEECE

Even Aristotle [1] paid attention to the necessity of explaining things by already explained and understood ones. Exposing the obligatory requirements for definitions he has noted: "A definition is introduced to make knowledge of something which is under consideration and we come to know it not by the first thing we meet but by a previous and the most famous one." The conclusion is that Aristotle was aware of the requirements for the definitions in order to make possible effective teaching and understanding.

Step by step, people dealing with Mathematics, noticed that many mathematical objects are defined uniquely by some of their properties only. For this reason, they began to use those properties in the corresponding descriptions. The other properties were expressed by phrases whose truthfulness was established by a subjected reasoning to certain rules. Thus, during the 5th century BC, theorems and their proofs came to birth. Here also the reasoning was subjected to the requirement of applying only words and assertions that had been used before in order to come to the condition characterized by "understanding". In Chapter 2 of the "Second Analytics" [1] Aristotle noted: "In order to make true the knowledge under proving the knowledge to prove by should also come out of truths, primary, direct, more famous and previous knowledge." Further he continued: "In this way, if we come to know the conclusion based on primary knowledge and consider this conclusion as true, then we know much more for the primary knowledge itself and consider it as more true than the conclusion is in fact…."

The requirements for definitions and proofs were stated by Aristotle not for Mathematics in particular, but for scientific knowledge in general. It is not known whether Euclid was acquainted with them but for sure he accepted them as an ideal and purpose in the structuring of mathematical knowledge. Since then, the requirements have become obligatory for all mathematicians. For example, in the XVII century Blaise Pascal (1623 – 1662) noted: "A real method consists of two main principles: the first one is whenever not to use a new notion before exposing its meaning clearly and the second one is whenever not to apply an assertion before proving it by already proved truths." Once already proved, corresponding assertions were used readily in the same manner as a wood manufacturer uses a tool bought or produced by himself. Thus, mathematicians like the wood manufacturer's economizing efforts, forces and time.

What can be seen from the above is that since the Ancient Greek period, three ideals for human activities have been imposed naturally in Mathematics, namely "understanding", "convicting" and "economizing". The motive to assure understanding gives birth to the requirement of defining by already defined notions. The motive to convict gives birth to the requirement of proving by already proved assertions. The motive to economize efforts, forces and time gives birth to the idea to use theorems readily without proving them each time. Thus, besides the “elemental, repeat and dependence” approach-triad a second approach-triad was formed even in Ancient Greece, i.e. the “understanding, convicting and economizing” approach-triad. Comparing pre-Greek Mathematics and the Ancient Greek one, the conclusion is that the second approach-triad is imposed during the Ancient Greek period. In connection with the “convicting” element we note that the following proposition is well-known in Mathematics History: “In the pre-Greek period the most important was the answer to the question “How”, while in the Ancient Greek period the most important was the answer to the question “Why”.” The conscious use of the first approach supports heuristic activities of people dealing with Mathematics and Didactics of Mathematics, while the conscious use of the second approach increases teaching activity efficiency in Mathematics and Didactics of Mathematics. In addition, in the case under consideration the difference between the two periods of development of Mathematics corresponds to two different periods of the intellectual development of Humanity generally speaking, namely: the period of naïve, non-critical and dogmatic belief, without-argument assertions of authorities, and the period of critical acceptance of with-argument assertions.

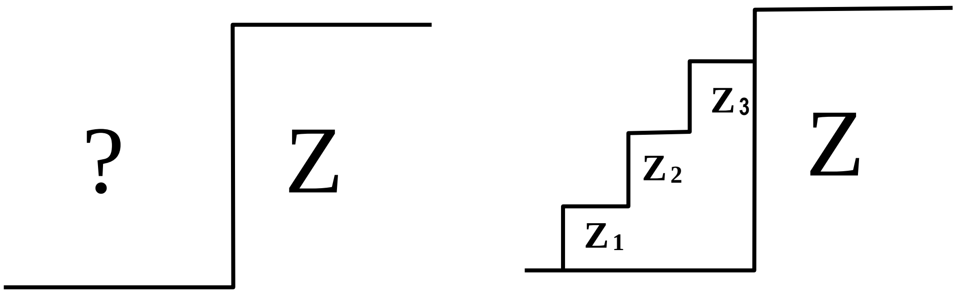
The ”elemental, repeat and dependence” approach-triad is implemented and used not always in full unification of the three components as stated. A repeat of the components themselves is possible, too. Usually, only the “elemental” is repeated as a phenomenon but in connection with other elements. This happens for example in the second rule of the “Descartes Method” [13]. It is well-known the statement: “Each of the difficulties under consideration should be divided into as many parts as it is needed and as it is possible to carry out aiming at the most efficient overcoming.” It is clear that here the question is about the phenomenon “elemental” itself in connection with different difficulties composed of different components.

4. THE “ELEMENTAL, REPEAT AND DEPENDENCE” APPROACH-TRIAD IN THE CONTEMPORARY DIDACTICS OF MATHEMATICS

Nowadays, a conscious application of difficulties division according to student possibilities is a reliable instrument for effectiveness of the teaching process, especially in Mathematics. This possibility is interwoven in the deductive structure of the content but also in the proof of every theorem and in the solution of every problem. It has been shown in [10] that every problem solution, theorem proof included, could be regarded as such a series of ordered problem solutions that contains only solutions of previous problems or is based on previously proved assertions. As a response of this situation, the notion of a problem-component has been introduced in [10]. The problem Z_k with a solution A_k is a problem-component of the problem Z_n with a solution A_n if the solution A_k is contained in A_n. This notion helps to enrich and sophisticate the language of the didactical theory of problems and Didactics of Mathematics as a consequence together with other ones connected with it like complexity of a problem solution, difficulty, didactical system of problems and s.o. By this possibility, the following conclusion is made in [10]:

"If the capabilities level of the students in a class is higher, then the teacher should choose exercise problems in such a way that less previously known solutions A_i of problems Z_i are included in the solution A_n of the problem Z_n. On the contrary, in case of a lower level, the previously known solutions should be more. This possibility should be used rationally to decrease difficulties in the proofs of complex theorems from the Mathematics curriculum."

It also follows that during exercises problems should be chosen and systemized in such a way that they should not only train already taught notions and concepts but should prepare the solutions of new problems and the proofs of new theorems. On the other hand, the approach amplifies the use of the visualness principle. This possibility was used in 1971 and the following picture appeared on the cover of [10]:



The step Z in the left figure represents an arbitrary problem with a complex unknown solution, while the steps Z_1, Z_2 and Z_3 in the right figure represent the solutions of the problem-components of Z. The proposed visual representations are in agreement with Rene Descartes' idea, which is exposed in his third rule for "a step by step learning up to a final knowledge of most complex subjects and phenomena". They are also in agreement with the more general idea of the Bulgarian academician Peter Kenderov for the so called "steps of knowledge" which have been used in the frames of the European project MATHEU (http://www.matheu.eu).

It is not difficult to conclude that the "dependence" element of the first approach-triad is an execution of Hegel's Dialectics law for the "common connection and related dependence". It turns out that the deductive structure of Mathematics influences essentially the activities sequence which assures "understanding" of mathematical knowledge and helps to work out capabilities for the exploration of this knowledge. In many cases the study of the influence itself leads to a general hypothesis that it is quite necessary to accept norms (requirements) for a forms structure of knowledge fixing in the Didactics of Mathematics. The approach should be deductive analogously to the deductive one which has been proposed by Aristotle for the structure of scientific knowledge in general. Aristotle's approach was cited in the above. According to the authors of the present paper, the norms concerning Didactics of Mathematics are the following:

1. The study of each capability connected with the use of a given knowledge should discover the factors on which this knowledge depends. According to the authors, the most essential factors are:

1.1. information about the elements of the corresponding knowledge and the relations between the elements;

1.2. information about the relations between the knowledge under consideration and other knowledge;

1.3. organization of suitable activities to exercise the use of the relations between the knowledge under consideration and other knowledge.

2. The study of each activity in Mathematics Education should distinguish the factors on which the learning of the activity under consideration depends and the factors which influence the learning of other related activities.

3. The structure and learning of each activities system should take account of the activities places. Each activity should be near sufficient number of already learned activities and known notions which are in relation with it.

4. The structure of each assertions system should take account of the assertions places. Each assertion should be near a sufficient number of already explained assertions. In other words, the structural studies should precede the relations ones.

According to the authors of the present paper, the acceptance of the listed norms (requirements) or similar to them, connected with the structure of Didactics knowledge, will assure a strong inner systematization but also it will direct the scientific investigations to a discovery of the links in activities, processes, phenomena, factors, etc. And it is well known that the knowledge about the link of an object with other objects is not less important for the object itself, for its elements and the relations among them. Without a pretention for a sufficient argumentation, a systematic and exhausted approach, some examples are pointed out in the sequel:

1. The capability of using a given mathematical knowledge depends essentially on:

a) its remembering duration;

b) its understanding (as a result and condition);

c) the executed activities in the learning of means for its exploitation in large.

2. Students' success in the learning of means for the exploitation in large of a given mathematical knowledge depends on:

a) the knowing of the knowledge as an independent entirety, as an element of another entirety but also on knowing of its structure;

b) the general mathematical background of teachers;

c) the background of teachers in Logics;

d) the background of teachers in Didactics;

e) the background of teachers in Psychology;

f) the level of preparation of students in Mathematics and on their attitude to Mathematic.

3. The remembering duration of a given knowledge depends on:

a) the level of its understanding (the understanding as a process and as a state);

b) the quantity and the character of the executed exercises during its learning;

c) the engagements degree of the activities during its learning;

d) the emotional circumstances during its learning;

e) the fatigue degree during its learning;

f) its maintenance during the teaching process.

4. The understanding of the knowledge as a result and as a state depends on:

a) its understanding as a process during its learning;

b) its maintenance as a result by a sufficient number of activities in which it is used.

5. The understanding of a given knowledge as a process during its learning depends on:

a) the level of knowing of previous knowledge involved in the learning;

b) the conditions to participate in activities which are connected with the learning;

c) the attention and the fatigue;

d) the speed of teacher's explanations and the conditions for the explanations to reach students;

e) the activity of students.

6. The activity of students in learning and using a given knowledge or capability depends on:

a) the understanding of the explanations in teaching and on students prior preparation for applications;

b) the conscious perception of the activity aim and the role of the knowledge or the capability;

c) the assured conditions for participation in the activity;

d) the emotional circumstances.

It is very fitting to represent the activities, the processes, the facts, etc. and the links in them by graph-schemes. Let us note that all the 6 examples from above could be detailed because any of the characteristics in a), b), c) and s.o. depends on other factors, processes, phenomena, activities, etc. The situation is similar to the one which concerns definitions of notions and proofs of assertions. Aristotle has called the last to be “disturbing infinity”. As it is well known, in Mathematics the “disturbing infinity” is being solved by accepting of the so called “elements” – preliminary concepts and axioms. A challenging problem is whether a successful exit concerning Didactics of Mathematics should be in the acceptance of some “elements”, too and if this is the case, then what should be the kind of these “elements”.

5. MATHEMATICAL MODELLING AS A STAGE IN THE DEVELEOPMENT OF HUMAN COGNITIVE ACTIVITY

It has been mentioned already above that Aristotle’s requirement to define by previously defined concepts and to prove by previously proved assertions concerns not only mathematical knowledge, but all sciences. More of this, we are going to show in the sequel that other essential features which are considered as specific for Mathematics only turn to be true for different domains of knowledge to one extent or another. What puts a peculiar label on the formation of capability for the discovery of a common structure (common properties, relations, etc.) of knowledge from different disciplines is the differentiation of sciences and as a consequence the differentiation of teaching disciplines. This happens even in the case when one of the disciplines (like Mathematics or Didactics of Mathematics) has undergone a conscious understanding and a reasonable back grounding to the degree of arbitrariness. It turns out that the formation of a capability under similar conditions which are characteristic for a given domain “ties” this capability only to knowledge from the same domain. The more the knowledge has a private character the more the link is strong. A capability “tie” concerns the conscious understanding to a degree of arbitrariness because according to the investigations of the psychologist L. Vygotsky, “ties” easily go to other domains. In [7] there is an example for a structure in the domain of the Russian language genders related to thinking which is a Descartes product of sets in fact. Let S_M be the set of the male gender nouns in the Russian language and O be the set of the endings for the male gender nouns in the Russian language. Then the nouns are elements of the product $S_M \times O$. There is an analogous situation with the verbs in Bulgarian, Russian and French. For example, if G stands for the Bulgarian regular verbs and O stands for the endings, then the different verbal forms are elements of the Descartes product $G \times O$. In fact G is divided into three classes of equivalence with respect to three characteristic vowels. Similarly, the set of the equations $ax = b$ could be divided into three classes with respect to the ways of solving which depend on the coefficients a and b. Another example is connected with the set of the construction problems defined by the following parametric one: “Construct a circle which is tangent to two given lines and is incident with a point not belonging to any of the lines." This problem is a particular case of the well known Apollonius Ancient Greek one.

In the last decades, the discovery and the use of examples like the considered ones, have shown that the non-conscious features of thinking structures could be made conscious and could be used in the support of understanding and in facilitating of the memory. Thus, transfer of structures in thinking from one domain to another becomes possible. For this purpose, a general approach is appropriate to the formation of higher psychic functions, using Vigotski's terminology [7]. It follows that the problem of the interdisciplinary connections should not be considered in a narrow pragmatic sense when the application of knowledge from one subject to another is rather elementary. The aim is to form higher psychic functions with a great degree of generality. The problem is connected directly with the mental development of students. Under the contemporary conditions in society, this leads to a natural domination of IT technologies in Education. It is quite understandable that the tendency is strongly apparent in Didactics of Mathematics. Being mathematicians, the professionals in Didactics of Mathematics dispose of the most powerful instrument for modeling, i.e. of Mathematics. For this reason, it is not by chance that for the first time the Law of qualitative and quantitative changes finds a good agreement with the accumulation of knowledge and this turns out to happen in Didactics of Mathematics [8]. The accumulation process takes place under certain conditions and it could be modeled by differential equations. The situation is similar to the modeling of ecologic phenomena [6] or the modeling of machinery amortization. Such an approach is used in the preparation of talented students. One of the results in this direction is the first place of the Bulgarian National Team in the country rankings of the International Mathematical Olympiad for secondary students in Japan, 2003. In the corresponding model, some ideas from Synergetics an Experimental Psychology are in essential connection with mathematical tools. Thus, it could be stated that the possibilities of the Dialectics laws are far from exhausting concerning the development of the sciences of teaching, Didactics of Mathematics included. What is important is to treat them in dependence on suitable mathematical approaches and tools. Note also, that Propaedeutics in Pedagogy corresponds to quantitative accumulations, while incite corresponds to qualitative ones in Psychology, i.e. incite is a jump in fact. Thus, the Law of qualitative and quantitative changes is executed. Some specialists in Didactics of Mathematics and History of Mathematics defend the thesis that in the deductive structure of knowledge in general and Mathematics in particular the so called principle of reasonable argumentation is secured as it is the case with the ideology of the slave-holding democracy in Society. As a result of its acceptance in Mathematics, a requirement is imposed to give answers to the question "Why".

6. COCLUSIONS

The considered examples, and not only they, show that Mathematics and Methodology of various teaching subjects, Didactics of Mathematics and Informatics included, have played and will play in the future an important and versatile role in the system of social sciences and as a consequence in the social practice, too. The role concerns not only technical devices for calculations even in case when using calculations one goes far beyond immediate observations and mechanical activities. Since Aristotle’s times, especially after the popularization of school education, the argumentation in knowledge exposition forms a proving style of thinking and heuristic capabilities at students’ age. People become critical and usually they accept only assertions which have established experimentally, by partial induction or by deduction and never on trust.

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