TWO REMARKABLE POINTS OF THE TRIANGLE GEOMETRY

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Abstract. *It is shown in the paper the discovery of two remarkable points of the triangle by means of "THE GEOMETER'S SKETCHPAD" software. Some properties of the points are considered too.*

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An effective way to establish geometrical relations is by the realization of experimental investigations on definite geometrical constructions. On the other hand, precise constructions are necessary for an efficient study which gives possibilities for exact observations on the dependences under examination. A good knowledge of the geometrical configuration and an appropriate instrument for elaboration are needed. A possible way is by the use of corresponding software. In the sequel it is demonstrated an application of "THE GEOMETER'S SKETCHPAD" (GSP) program and of two well known theorems to the discovery of two remarkable points of the triangle plane geometry. The discovery is based on the following two theorems which are proved in [1]:

Theorem 1. *Let k be the circumcircle of ΔABC and the circle k_c with radius ρ_c be internally tangent to k and to the sides CA and CB. If r is the radius of the incircle ΔABC) and $\angle BCA = \gamma$, then $\rho_c = \dfrac{r}{\cos^2 \frac{\gamma}{2}}$ (Fig. 1).*

Theorem 2. *Let k be the circumcircle of ΔABC and the circle k'_c with radius ρ'_c be externally tangent to k and to the lines CA and CB. If r_c is the radius of the externally tangent circle of ΔABC with respect to AB and $\angle BCA = \gamma$, then $\rho'_c = \dfrac{r_c}{\cos^2 \frac{\gamma}{2}}$ (Fig. 2).*

The usual notations for the sides and the angles of a given ΔABC are be used. Additionally, the tangent points of k_c with CA and CB are denoted by C_a and

C_b, respectively (Fig. 1), while the tangent points of k'_c with CA and CB are denoted by C'_a and C'_b, respectively (Fig. 2).

As stated in [2], it follows the following property from theorem 1:

Property 1. *The points C_a and C_b together with the incenter I of ΔABC are collinear (Fig. 1).*

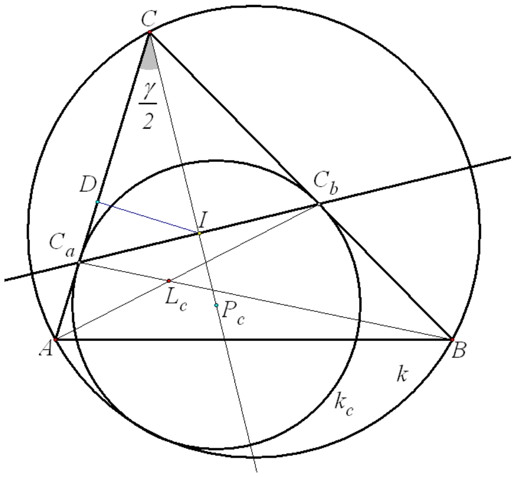

Fig. 1

The proof could be deduced by means of the equalities $CI = \dfrac{DI}{\sin\frac{\gamma}{2}} = \dfrac{r}{\sin\frac{\gamma}{2}}$ and $CC_a = C_a P_c ctg\frac{\gamma}{2} = \rho_c ctg\frac{\gamma}{2}$, which follow from the triangles IDC and $P_c C_a C$, respectively (Fig. 1). The equalities, together with theorem 1, lead to the conclusion that $\angle CIC_a = 90°$. Thus, CI is angular bisector and altitude of $\Delta C_a C_b C$ from the vertex C. Consequently, the points I, C_a and C_b are collinear, the point I being the midpoint of the segment $C_a C_b$ (Fig. 1).

Analogously, it follows from theorem 2:

Ptoperty 2. *The points C'_a and C'_b, together with the center I_c of the externally tangent circle of ΔABC with respect to AB are collinear (Fig. 2).*

It could be deduced from corollaries 1 and 2 an easy way for the construction of the points C_a, C_b, C'_a and C'_b. Such an observation facilitates the realization of all necessary constructions by GSP.

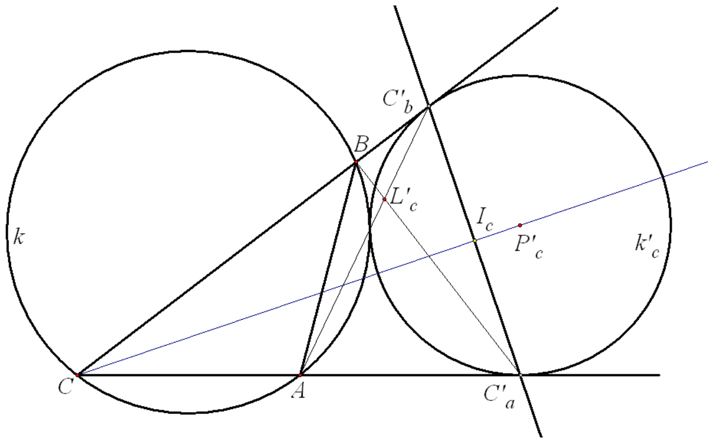

Fig. 2

Let $L_c = AC_b \cap BC_a$ (Fig. 1) and $L'_c = AC'_b \cap BC'_a$ (Fig. 2). The points L_a, L_b, L'_a and L'_b are determined analogously. Some observations by GSP on the relations of the points with the vertices of ΔABC give arguments for the formulation of the following two properties:

Property 3. *The lines* AL_a, BL_b *and* CL_c *are concurrent in the point* T.

Property 4. *The lines* AL'_a, BL'_b *and* CL'_c *are concurrent in the point* T'.

Barycentric coordinates with respect to ΔABC like $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$ could be applied to the proof of the above properties as well as of the next ones. What is used for the determination of the coordinates of the points C_a and C_b is that $I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)$ [3, p. 91], where $p=\frac{a+b+c}{2}$ and also the formula for scalar product of vectors [3, p. 60] in the equation $\overrightarrow{CI}.\overrightarrow{C_bI}=0$ (it follows from property 1). Thus we get: $C_a\left(\frac{a}{p},0,\frac{p-a}{p}\right)$, $C_b\left(0,\frac{b}{p},\frac{p-b}{p}\right)$.

It is obtained analogously that $C'_a\left(\frac{a}{p},0,-\frac{p-b}{p-c}\right)$, $C'_b\left(0,\frac{b}{p},-\frac{p-a}{p-c}\right)$.

Using the coordinates of the points C_a, C_b, C'_a and C'_b, we determine the equations of the pairs of lines AC_b, BC_a and AC'_b, BC'_a. Next, their common points L_c and L'_c are determined in the form:

$$(1)\qquad L_c\left(\frac{a(p-b)}{p^2-ab},\frac{b(p-a)}{p^2-ab},\frac{(p-a)(p-b)}{p^2-ab}\right),$$

$$(2)\qquad L'_c\left(-\frac{a(p-a)}{(p-c)^2-ab},-\frac{b(p-b)}{(p-c)^2-ab},\frac{(p-a)(p-b)}{(p-c)^2-ab}\right).$$

It is well known that three points $M(x,y,z)$, $M_1(x_1,y_1,z_1)$ and $M_2(x_2,y_2,z_2)$ are collinear iff the following equality is verified:

$$(3)\qquad \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0 \text{ [3, p. 61]}.$$

Now, by (1) and (3) the equation of the line CL_c is obtained in the form: $CL_c : b(p-a)x-a(p-b)y=0$. Substitute the equations $p-a=rctg\frac{\alpha}{2}$, $p-b=rctg\frac{\beta}{2}$, $a=2R\sin\alpha$ and $b=2R\sin\beta$. The last equation takes the form $CL_c : \sin^2\frac{\beta}{2}x-\sin^2\frac{\alpha}{2}y=0$.

Analogously we obtain the equations $AL_a : \sin^2\frac{\gamma}{2}y-\sin^2\frac{\beta}{2}z=0$ and $BL_b : \sin^2\frac{\alpha}{2}z-\sin^2\frac{\gamma}{2}x=0$. Next, it is easy to check that the three equations are verified by the coordinates of the point

$$(4)\qquad T\left(\frac{\sin^2\frac{\alpha}{2}}{\tau},\frac{\sin^2\frac{\beta}{2}}{\tau},\frac{\sin^2\frac{\gamma}{2}}{\tau}\right),$$

where $\tau=\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}$.

Analogously, by means of (2) and (3) it follows that the lines AL'_a, BL'_b and CL'_c pass through the point

(5) $$T'\left(\frac{\cos^2\frac{\alpha}{2}}{\tau'},\frac{\cos^2\frac{\beta}{2}}{\tau'},\frac{\cos^2\frac{\gamma}{2}}{\tau'}\right),$$

where $\tau'=\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}$.

This ends the proof of properties 3 and 4.

Argumentation for the points T and T' thus obtained to be considered as remarkable of ΔABC could be found in some interesting properties of them. The search is possible by GSP again.

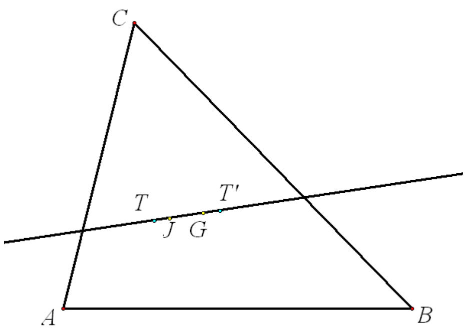

Fig. 3

A first observation is connected with the following:

Property 5. *T and T' are in-points of ΔABC (Fig. 3).*

The proof of this property could be deduced directly from (4) and (5).

A search of a relation between the points T, T' and classic remarkable points of ΔABC leads to the following:

Property 6. *The center of gravity G and the Gergonne point J of ΔABC are on the line TT' (Fig. 3).*

The proof of this property could be obtained by the coordinate representations of G and J: $G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $J\left(\frac{tg\frac{\alpha}{2}}{\theta},\frac{tg\frac{\beta}{2}}{\theta},\frac{tg\frac{\gamma}{2}}{\theta}\right)$, where $\theta=tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}$ [3, p. 97]. The validity of (3) for the triples of points T, T', G and T, T', J could be verified from (4) and (5) by substitution.

By the help of GSP relations of the points T and T' under known transformations in the plane of ΔABC could be found. Thus, a dependence exists between T and T' under isogonal transformation which could be formulated in the following way:

Property 7. *The points T and T' are isogonal conjugate with respect to ΔABC.*

It could be used in the proof of the above property that the isogonal conjugate of a given point $P(x, y, z)$ is the point $Q\left(\frac{a^2}{xt}, \frac{b^2}{yt}, \frac{c^2}{zt}\right)$, where $t = \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}$ [3, p. 65]. It follows easily from (4) and (5) that the same dependence exists between the points T and T' which implies that they are isogonal conjugate.

On the grounds of the obtained properties of the points T and T' we call them remarkable points of ΔABC.

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