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BOUNDS FOR THE UNIQUE POSITIVE ROOT OF A POLYNOMIAL IN MATHEMATICS OF FINANCE[1]

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Abstract. In this paper we obtain some lower bounds for the unique positive root (the effective rate of the bond, or effective rate of an annuity) of the algebraic equation $x^n - \sum_{i=1}^{n} a_i x^{n-i} = 0$. This is a further improvement of the known results in the financial mathematics. Interesting numerical examples are presented.

Key words: upper bounds for positive roots of algebraic polynomials, spectral radius, effective rate of the bond, internal rate of return of investment, cash flow streams of the investment

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1. Introduction

Several problems in the classical mathematics of finance lead to a class of polynomial equation [3]:

$$P(x) = Cx^n - \sum_{j=1}^{n-1} B_j x^{n-j} - A = 0 \tag{1}$$

with only one single positive root σ, where A is the purchase price of the bond paid to the issuer; $B_j \equiv B$ is a periodic payment paid according to the contract

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rate of the bond; $C = K + B$, where K is the purchase rate of the bond when sold at the bond market; n is the term of the bond (usually number of full years or months).

The effective rate of the bond ρ is given by the unique positive root σ of equation (1) (according to DESCARTES' rule of sign) and $\rho = \frac{1}{\sigma} - 1$.

For ordinary annuities with growing periodic payments we have: C is the present value of the annuity; $B_j = h^{j-1}B$ is the periodic payment beginning with the amount of B and with growth factor h [9].

In the classical mathematics of finance one important measure for the profitability of an investment is its internal rate of return.

The internal rate of return is an abstract number associated with the cash flow streams of the investment.

For investment form the associated polynomial equation determining the internal rate of return is

$$P(x) = x^n - \sum_{i=1}^{n} a_i x^{n-i} = 0. \tag{2}$$

Various estimations for the unique root σ of equation (2) can be found in [2], [6], [8], [7] and [4].

In this paper we give another estimations.

The following theorems by Deutsch [1] are more often applicable:

Theorem A. Let $A = (a_{ij})$ be a non-negative irreducible $n \times n$ matrix and let $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ be positive vectors satisfying

$$Ax = Dx,$$

$$A^T y = Dy,$$

for some positive diagonal matrix

$$D = diag(d_1, d_2, \ldots, d_n).$$

If x is not Perron vector of A, then

$$\rho(A) > \frac{y^T D x}{y^T x}, \tag{3}$$

where $\rho(A)$ is the spectral radius of the matrix A.

Theorem B. Using the same notation as in Theorem A, then

$$\rho(A) \geq \prod_{i=1}^{n} d_i^{\frac{d_i x_i y_i}{y^T D x}} \geq \frac{y^T D x}{y^T x}. \tag{4}$$

Theorem C. Using the same notation as in Theorem A, then

$$\rho(A) \geq t - \prod_{i=1}^{n} (t - d_i)^{\frac{x_i y_i}{y^T x}} \geq \frac{y^T D x}{y^T x}, \tag{5}$$

for all

$$t > \rho(A) + \max_{1 \leq i \leq n} (d_i - a_{ii}).$$

Remark 1. $t = \beta + \max\limits_{1 \leq i \leq n} (d_i - a_{ii})$ is a lower bound for $\rho(A)$.

Some of the possible values of β are:

a) the largest row sum of A;
b) the largest column sum of A;
c) $\max_i d_i$.

2. Main results

We note that, the known estimations for the positive root σ, based on Theorems A - C are valid for $x = (1, 1, \ldots, 1)^T$.

The first estimation for σ, based on Theorem B with arbitrary $x_i = \lambda^i$, $i = 1, \ldots, n$, $\lambda > 0$ can be found in [4]:

$$\sigma \geq \frac{1}{\lambda} (q(\lambda))^{\frac{q(\lambda)}{\lambda q'(\lambda)}}, \tag{6}$$

where

$$q(\lambda) = \sum_{i=1}^{n} a_i \lambda^i.$$

Evidently,

$$0 = P(\sigma) = \sigma^n - \sum_{i=1}^{n} a_i \sigma^{n-i} = \sigma^n \left(1 - q\left(\frac{1}{\sigma}\right)\right),$$

i.e.

$$q\left(\frac{1}{\sigma}\right) = 1.$$

Remark 2. Consequently, if we choos λ to be near $\frac{1}{\sigma}$, then the estimation (6) will be more precisely.

In this paper we give estimations for σ, based on Deutsch's Theorem C for arbitrary positive vector x.

We state the following

Theorem 1. Let

$$a_k \geq 0,\ k = 2, \dots n,\ \sum_{i=1}^{n} a_i x_i \geq \left(\frac{x_1^n}{x_n}\right)^{\frac{1}{n-1}}.$$

For arbitrary $x_1, \dots, x_n$, for the positive root σ of the equation (2) the following estimation hold:

$$\sigma \geq a_1 + \tfrac{2}{x_1}\sum_{i=2}^{n} a_i x_i - \left(\tfrac{1}{x_1}\sum_{i=2}^{n} a_i x_i\right)^{\frac{x_1}{x_1 + \sum_{i=2}^{n} \mu_i a_i x_i}}$$

(7)
$$\times \left(a_1 + \tfrac{2}{x_1}\sum_{i=2}^{n} a_i x_i - \tfrac{x_1}{x_2}\right)^{\frac{\frac{x_2}{x_1}\sum_{i=2}^{n} a_i x_i}{x_1 + \sum_{i=2}^{n} \mu_i a_i x_i}}$$

$$\times \dots \times \left(a_1 + \frac{2}{x_1}\sum_{i=2}^{n} a_i x_i - \frac{x_{n-1}}{x_n}\right)^{\frac{\frac{x_n}{x_{n-1}}\sum_{i=n}^{n} a_i x_i}{x_1 + \sum_{i=2}^{n} \mu_i a_i x_i}},$$

where

$$\mu_i = \sum_{k=1}^{i-1} \frac{x_{k+1}}{x_k}, \;\; i = 2, 3, \ldots, n.$$

Proof. Let us associate with the polynomial $p(x)$ the corresponding matrix

$$A^{(n)}_{a_1,\ldots,a_n} = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

with $\det(A^{(n)}_{a_1,\ldots,a_n} - xI) = (-1)^n P(x)$.

The matrix $A^{(n)}_{a_1,\ldots,a_n}$ is non-negative and irreducible. By the Perron-Frobenius theorem, this implies that $A^{(n)}_{a_1,\ldots,a_n}$ has a positive eigenvalue equal to its spectral radius $\rho(A^{(n)}_{a_1,\ldots,a_n})$, i.e. $\rho(A^{(n)}_{a_1,\ldots,a_n}) = \sigma$.

Let $A = A^{(n)}_{a_1,\ldots,a_n}$. From the relation

$$Ax = Dx,$$

we find

$$d_1 = \frac{1}{x_1} \sum_{i=1}^{n} a_i x_i; \;\; d_k = \frac{x_{k-1}}{x_k}, \;\; k = 2, 3, \ldots, n.$$

Then the system

$$(A^T - D)y = 0$$

yields

$$y_1 = 1, \quad y_k = \frac{1}{x_{k-1}} \sum_{i=k}^{n}, \;\; k = 2, 3, \ldots, n.$$

Let (see, Theorem C)

$$t = \beta + \max_{1 \le i \le n} (d_i - a_i) = \max_i d_i + \max_{1 \le i \le n} (d_i - a_i).$$

Using the assumptions in Theorem 1 we have

$$d_1 - a_1 = \max_{1 \le i \le n} (d_i - a_i), \;\; d_1 = \max_i d_i$$

and

$$t = a_1 + \frac{2}{x_1} \sum_{k=2}^{n} a_k x_k. \tag{8}$$

Now, we have

$$y^T x = x_1 + \sum_{i=2}^{n} \mu_i a_i x_i \tag{9}$$

and

$$x_1 y_1 = x_1, \quad x_k y_k = \frac{x_k}{x_{k-1}} \sum_{i=k}^{n} a_i x_i, \;\; k = 2, 3, \ldots, n. \tag{10}$$

Evidently

$$\begin{aligned}
t - d_1 &= \tfrac{1}{x_1} \sum_{k=2}^{n} a_k x_k, \\
t - d_2 &= a_1 + \tfrac{2}{x_1} \sum_{k=2}^{n} a_k x_k - \tfrac{x_1}{x_2}, \\
&\vdots \\
t - d_n &= a_1 + \tfrac{2}{x_1} \sum_{k=2}^{n} a_k x_k - \tfrac{x_{n-1}}{x_n}.
\end{aligned}$$

From the last expression, eq. (8), (9), (10) and from the left inequality in (5) we get the bound (7).

The proof is complete.

3. Special choise of the parameters

The vector $x = (x_1, x_2, \ldots, x_n)^T > 0$ can be taken in arbitrary way.

First, if choose

$$x_i = \lambda^i, \quad i = 1, 2, \ldots, n, \ \ \lambda > 0$$

we get the following result.

Corollary 1. For arbitrary positive λ for the positive root σ of the polynomial $P(x)$ the following estimation hold:

$$\sigma \geq a_1 + 2G - \sqrt[F]{G\left(a_1 + 2G - \frac{1}{\lambda}\right)^{F-1}} \tag{11}$$

where

$$G = \sum_{i=2}^{n} a_i \lambda^{i-1},$$

$$F = a_1 + \sum_{i=2}^{n} (i-1) a_i \lambda^i.$$

We note that the lower bound for σ is more precise.

Remark 3. Consequently, if we choos λ to be near $\frac{1}{\sigma}$, then the estimation (11) will be more precisely.

Another special choose for x_i be the following

$$x_{i_k} = \frac{a_{i_k}^{\mu_k - 1}}{a_{i_1}^{\mu_1} + a_{i_2}^{\mu_2} + \cdots + a_{i_m}^{\mu_m}}, \quad k = 1, 2, \ldots, m,$$

where $0 < x_i$ are arbitrary for $i \neq i_1, i_2, \ldots, i_m$, and $\mu_1, \mu_2, \ldots, \mu_m$ are completely numbers.

Remark 4. The explicit bounds for the unique positive root in the term of finance can be obtained using the approach given in this paper.

Remark 5. Such results are important for the determination of the R-order of convergence of iterative process (IP) that produces a sequence of approximations $\{x^k\}$ with the limit point x^*.

For the errors

$$\eta^{(k+1)} \leq \gamma \prod_{i=0}^{n-1} \left(\eta^{(k-i)}\right)^{a_{i+1}},$$

$$a_i \geq 0,\ i = 1, 2, \ldots, n,\ \gamma > 0,\ k > n - 1.$$

The recurrence has the R-order of convergence $O_R(IP, x^*)$ of at least σ, where σ is the unique positive root of the equation (2) (see, [5]).

4. Numerical example

The polynomial

$$P(x) = x^3 - x^2 - 2x - 3 = 0$$

has the root $\sigma \approx 2.37442$.

We observe that in the case $x_i \equiv 1$ from the known estimation by M.Petkovic and L. Petkovic [8]:

$$\sigma \geq \frac{y^T D x}{y^T x} = \frac{\sum_{i=1}^{n} i a_i}{1 + \sum_{i=2}^{n} a_i},$$

we find

$$\sigma \geq \frac{7}{3} = 2.3(3).$$

From (6) with $\lambda = \frac{1}{2}$ we get the following bound:

$$\sigma \geq 2\,(1.375)^{0.523809523809523809 52} \approx 2.36305737.$$

From the new bound (11) with $\lambda = \frac{1}{2}$ we find:

$$\sigma \geq 4.5 - (5.50127125486284241139)^{0.44444444444444444444} \approx 2.36648994.$$

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ПРЕЦИЗНИ ОЦЕНКИ ЗА ЕДИНСТВЕНИЯ ПОЛОЖИТЕЛЕН КОРЕН НА ЕДИН КЛАС ПОЛИНОМИ ОТ ОБЛАСТТА НА ФИНАНСОВАТА МАТЕМАТИКА

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Резюме. Редица класически проблеми от финансовата математика водят до изследване и получаване на прецизни оценки за нули на алгебричен полином, което е в тясна връзка с оценки за ефективен размер на бон с нарастваща премия и участие на вторичен пазар. Получените в настоящата статия оценки могат да се използват и за оценка на инвестиционни проекти и риск на възвръщаемост, както и за прецизиране на реда на сходимост на редица итерационни методи от алгебрата и анализа.