ПЛОВДИВСКИ УНИВЕРСИТЕТ "ПАИСИЙ ХИЛЕНДАРСКИ", БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 37, КН. 3, 2010 – МАТЕМАТИКА
PLOVDIV UNIVERSITY "PAISSII HILENDARSKI", BULGARIA
SCIENTIFIC WORKS, VOL. 37, BOOK 3, 2010 – MATHEMATICS

CERTAIN CLASSES OF FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The aim of this paper is to obtain coefficient estimates, distortion theorem, and radii of close-to-convexity, starlikeness and convexity for functions belonging to the subclass $S_T(n, \alpha, \beta)$ with negative coefficients.

Key words: univalent, convex, starlike
Mathematics Subject Classification 2000: 30C45

1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let S^* and C be subclasses of S that are, respectively, starlike and convex.

A function

$$f(z) = \widetilde{C} \iff \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.2}$$

Let S_p be a class of starlike functions related to $\widetilde{C}$ defined as

$$f(z) \in S_p \iff \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U. \tag{1.3}$$

Note that

$$f \in \widetilde{C} \iff zf'(z) \in S_p. \tag{1.4}$$

A function f of the form (1.1) is in $S_p(\alpha)$ if it satisfies the analytic characterization:

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad -1 \leq \alpha < 1,\ z \in U. \tag{1.5}$$

The function $f \in \widetilde{C}(\alpha)$ if and only if $zf'(z) \in S_p(\alpha)$.

By $\widetilde{C}_\beta$, $0 \leq \beta < \infty$ we denote the class of all β-convex functions introduced by Kanas and Wisniowska [1]. It is know that [1] $f \in \widetilde{C}_\beta$ if and only if it satisfies the following condition:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf'(z)}{f'(z)}\right|, \quad z \in U,\ \beta \geq 0. \tag{1.6}$$

We consider the class S^*_β, $0 \leq \beta < \infty$, of β-starlike functions [2], which are associated with the class $\widetilde{C}_\beta$ by the relation

$$f \in C^*_\beta \iff zf'(z) \in S^*_\beta. \tag{1.7}$$

Thus, the class S^*_p is the subclass of S, consisting of functions that satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U,\ \beta \geq 0. \tag{1.8}$$

For a function $f \in S$, we define

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= \frac{f(z) + zf'(z)}{2} = Df(z) \\ D^n f(z) &= D(D^{n-1} f(z)), \qquad n \in \mathbb{N} = \{1, 2, \dots\} \end{aligned} \tag{1.9}$$

It can be easily seen that

$$D^n f(z) = z + \sum_{n=2}^{\infty} \left(\frac{k+1}{2}\right)^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.10}$$

For $\beta \geq 0$, $-1 \leq \alpha \leq 1$ and $n \in \mathbb{N}_0$ let $S(n, \alpha, \beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic condition

$$\Re\left\{\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right\} > \beta\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right|. \tag{1.11}$$

We denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.12}$$

Further, we define the class $S_T(n, \alpha, \beta)$ by

$$S_T(n, \alpha, \beta) = S(n, \alpha, \beta) \cap T. \tag{1.13}$$

2. Coefficient estimates

Theorem 2.1. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.12) *to be in the class $S_T(n, \alpha, \beta)$ is that*

$$\sum_{k=1}^{\infty} [k(1+\beta) - (\alpha+\beta)] \left(\frac{k+1}{2}\right)^n a_k \leq 1 - \alpha \tag{2.1}$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$ and $n \in \mathbb{N}_0$.

Proof. Let (2.1) holds true, then we have

$$\beta\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \Re\left\{\frac{z(D^n f(z))'}{D^n f(z)} - 1\right\} \leq (1+\beta)\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right|$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)\left(\frac{1+k}{2}\right)^n |a_k|}{1 - \sum_{k=2}^{\infty}\left(\frac{1+k}{2}\right)^n |a_k|} \leq 1 - \alpha.$$

Then $f(z) \in S_T(n, \alpha, \beta)$.

Conversely, let $f(z) \in S_T(n,\alpha,\beta)$ and z be real, then

$$\frac{1-\sum_{k=2}^{\infty} k\left(\frac{k+1}{2}\right)^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{1+k}{2}\right)^n a_k z^{k-1}} - \alpha \geq \beta \left| \frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{k+1}{2}\right)^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_k z^{k-1}} \right|,$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality (2.1).

Remark 1. If $f(z) \in S(n,\alpha,\beta)$ the condition (2.1) is only sufficient.

Remark 2. Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then

$$a_k \leq \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\left(\frac{k+1}{2}\right)^n}, \qquad k \geq 2. \tag{2.2}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{[k(1+\alpha)-(\alpha+\beta)]\left(\frac{k+1}{2}\right)^n}\, z^k. \tag{2.3}$$

3. Growth and distortion theorems

Theorem 3.1. Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then

$$|D^i f(z)| \geq |z| - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.1}$$

and

$$|D^i f(z)| \leq z + \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.2}$$

for $z \in U$, where $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^n z^2 \tag{3.3}$$

Proof. Note that $f(z) \in S_T(n, \alpha, \beta)$ if and only if $D^i f(z) \in S_T(n, \alpha, \beta)$ and that

$$D^i f(z) = z - \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^i a_k z^k. \tag{3.4}$$

Using Theorem 2.1 we know that

$$(2 - \alpha + \beta) \left(\frac{3}{2}\right)^{n-i} \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^i a_k \leq 1 - \alpha \tag{3.5}$$

that is, that

$$\sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^i a_k \leq \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^{n-i}. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$|D^i f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^i a_k \geq |z| - \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^{n-i} |z|^2 \tag{3.7}$$

and

$$|D^i f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^i a_k \leq |z| + \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^{n-i} |z|^2 \tag{3.8}$$

Finally, we note that the bounds in (3.1) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^{n-i} z^2. \tag{3.9}$$

This completes proof of Theorem 3.1.

Corollary 3.1. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n, \alpha, \beta)$. Then*

$$|z| - \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^n |z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{2-\alpha+\beta} \left(\frac{2}{3}\right)^n |z|^2. \tag{3.10}$$

The equalities in (3.10) *are attained for the function $f(z)$ given by* (3.3).

Proof. Taking $i = 0$ in Theorem 2.1, we immediately obtain (3.10).

4. Radii of close-to-convexity, starlikeness and convexity

A function $f(z) \in T$ is said to be close-to-convex of order ρ if it satisfies

$$\Re f'(z) > \rho, \quad 0 \le \rho < 1, \; z \in U. \tag{4.1}$$

Theorem 4.1. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n, \alpha, \beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_1$ where*

$$r_1 = r_1(n, \alpha, \beta, \rho) = \inf_k \left\{ \frac{(1-\rho)[k(1+\beta) - (\alpha+\beta)](k+1)^n}{2^n k(1-\alpha)} \right\}^{\frac{1}{k-1}}, \quad k \ge 2 \tag{4.2}$$

The result is sharp, with extremal $f(z)$ given by (2.3).

Proof. We must show that $|f'(z) - 1| \le 1 - \rho$ for $|z| < r_1(n, \alpha, \beta, \rho)$ where $r_1(n, \alpha, \beta, \rho)$ is given by (4.2). Indeed we find from (1.12) that

$$|f'(z) - 1| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \le 1 - \rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \le 1. \tag{4.3}$$

But, by Theorem 2.1, (4.3) will be true if

$$\left(\frac{k}{1-\rho} \right) |z|^{k-1} \le \frac{[k(1+\beta) - (\alpha+\beta)](k+1)^n}{2^n(1-\alpha)}$$

that is, if

$$|z| \le \left\{ \frac{(1-\rho)[k(1+\beta) - (\alpha+\beta)](k+1)^n}{k(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.4}$$

Theorem 4.1 follows easily from (4.4).

Theorem 4.2. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n,\alpha,\beta)$. Then the function $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.5}$$

The result is sharp, with the extreme function $f(z)$ given by (2.3).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\rho \quad \text{for} \quad |z| < r_2(n,\alpha,\beta,\rho)$$

where $r_2(n,\alpha,\beta,\rho)$ is given by (4.5). Indeed we find again from (1.12) that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\rho}{1-\rho} \right) a_k |z|^{k-1} \le 1 \tag{4.6}$$

But, by Theorem 2.1, (4.6) will be true if

$$\left(\frac{k-\rho}{1-\rho} \right) |z|^{k-1} \le \frac{[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(1-\alpha)2^n},$$

that is, if

$$|z| \le \left\{ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.7}$$

Corollary 4.1. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n,\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$(4.8)\quad r_3 = r_3(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{k(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2.$$

The result is sharp with extremal function $f(z)$ given by (2.3).

References

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Received 09 July 2010

НЯКОИ КЛАСОВЕ ОТ ФУНКЦИИ С ОТРИЦАТЕЛНИ КОЕФИЦИЕНТИ

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Резюме. Целта на тази статия е да се получат коефициентни оценки, теореми за ръста и радиусите на почти изпъкналост, звездност и изпъкналост за функциите приндалежащи на класа $S_T(n,\alpha,\beta)$ с отрицателни коефициенти.