ПЛОВДИВСКИ УНИВЕРСИТЕТ "ПАИСИЙ ХИЛЕНДАРСКИ", БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 37, КН. 3, 2010 – МАТЕМАТИКА
PLOVDIV UNIVERSITY "PAISSII HILENDARSKI", BULGARIA
SCIENTIFIC WORKS, VOL. 37, BOOK 3, 2010 – MATHEMATICS

ON THE UNIT GROUPS OF COMMUTATIVE MODULAR GROUP ALGEBRAS

Velika N. Kuneva, Todor Zh. Mollov

Abstract. Let G be an abelian group and R be a commutative ring with identity of prime characteristic p. Denote by RG the group algebra of G over R, by $V(RG)$ the group of normalized units in RG, by G_p the p-component of G, by tG the torsion subgroup of the group G and by R_p^* the p-component of the unit group R^* of the ring R. We prove that if G is a direct factor of $V(RG)$, then $V(RG)/G$ is a p-group if and only if the pair (R, G) satisfies exactly one of the following conditions (*):

1) $G = G_p$;

2) $G \neq G_p, tG = G_p$ and the ring R is indecomposable;

3) $p = 3, R^* = \langle -1 \rangle \times R_3^*, G = A \times G_3, |A| = 2$ and

4) $p = 2, R^* = R_2^*, G = A \times G_2, |A| = 3$ and the equation $X^2 + XY + Y^2 = 1 + N(R)$ has only the trivial solutions in the quotient ring $R/N(R)$, where $N(R)$ is the nil-radical of R.

Let R be a direct product of m commutative perfect rings R_i and let G be a direct factor of $V(R_iG)$, i=1,2,...m. Then we give a complete description, up to isomorphism,

(i) of the maximal divisible subgroup of $V(RG)$ if every pair (R_i, G), i=1,2,...m, satisfies exactly one of the conditions $(^*)$ and

(ii) of $V(RG)$ if $V(R_iG)/G$, i=1,2,...m, are simply presented p-groups and the ring R is without nilpotent elements.

Key words: commutative modular group algebras, group of normalized units, Ulm-Kaplansky invariants, maximal divisible subgroup, simply presented abelian groups, p-mixed abelian groups

Mathematics Subject Classification 2000: 16S34, 16U60, 20C05, 20C07

1. Introduction

Let RG be the group algebra of an abelian group G over a commutative ring R with identity. Denote by $U(RG)$ the multiplicative group of RG and by $S(RG)$ the Sylow p-subgroup of the group $V(RG)$ of normalized units in RG, that is the p-component of $V(RG)$. The investigations of the group $S(RG)$ begin with the fundamental papers of Berman (1967a and 1967b) in which a complete description of $S(RG)$, up to isomorphism, is given when G is a countably infinite abelian p-group and R is a countable field of characteristic p such that if G is not a restricted direct product of cyclic groups, then the field R is perfect. Further Mollov (1977 and 1981) calculates the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(RG)$ when G is an arbitrary abelian group and R is a field of positive characteristic p. Let R be a commutative ring with identity of prime characteristic p. Nachev and Mollov (1980) calculate the invariants $f_\alpha(S)$ with the only restriction G to be an abelian p-group. Nachev (1995) calculates the invariants $f_\alpha(S)$ without restrictions on G. Moreover, in all indicated cases the authors give a full description, up to isomorphism, of the maximal divisible subgroup of $S(RG)$.

Let G be an abelian p-group and let K be a perfect field of characteristic p. May (1988) proves that $S(KG)$ is simply presented if and only if G is a simply presented abelian p-group. Therefore, if G is a simply presented abelian p-group, then the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(KG)$ together with the description of the maximal divisible subgroup of $S(KG)$ give a full description, up to isomorphism, of the group $S(KG)$.

Kuneva (2006) gives a description of the maximal divisible subgroup of $V(RG)$ when G is a p-mixed abelian group and the ring R is a direct product of n commutative indecomposable rings with identity of characteristic $p, n \in \mathbb{N}$.

Using a result of May, Mollov and Nachev (2010) and May (2008), Mollov and Nachev (2010a) give a full description, up to isomorphism, of the group $V(RG)$ when R is a direct product of m perfect fields of characteristic p, $m \in \mathbb{N}$, G is a p-mixed abelian group, G_p is simply presented and either (i) G is splitting or (ii) G is of countable torsion free rank.

The present paper continues the mentioned investigations of Mollov and Nachev (2010a) of $V(RG)$.

2. Some concepts and preliminary results

We recall some well known definitions. Let G be an abelian group and p be a prime. The group G is called p-mixed if the torsion subgroup of G is

p-primary. We use the signs $\coprod$ and Π for a mark of a coproduct of groups and a direct product of groups (algebras), respectively. Denote by $\coprod_{\alpha} G$ a coproduct of α copies of G, where α is a cardinal number. It is not hard to see that if α is an ordinal, then $(G_p)^{p^\alpha} = (G^{p^\alpha})_p$. Hence we can denote $G_p^{p^\alpha} = (G_p)^{p^\alpha} = (G^{p^\alpha})_p$.

The abelian group terminology is in agreement with the books of Fuchs (1970 and 1973).

Let R be a commutative ring with identity. We denote $R(p) = \{r \in R | r^p = 0\}$. The group algebra RG is called modular if the characteristic of R is a prime number p. The following results are well known.

Lemma 2.1. (Kuneva, Mollov and Nachev, 2009) *If G is an abelian group and R is a commutative perfect ring with identity of prime characteristic p, then G is p-balanced in $V(RG)$.*

The following result is due to Nachev (1995).

Theorem 2.2. *If R is a commutative ring with identity of prime characteristic p, G is an abelian group and α is the first ordinal with the property $R^{p^\alpha} = R^{p^{\alpha+1}}$ and $G^{p^\alpha} = G^{p^{\alpha+1}}$, then*

$$dS(RG) \cong \coprod_{\lambda} \mathbb{Z}(p^\infty),$$

where

1) $\lambda = max(|R^{p^\alpha}|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} \neq 1$;

2) $\lambda = max(|R^{p^\alpha}(p)|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} = 1, G^{p^\alpha} \neq 1$ and $R^{p^\alpha}(p) \neq 0$ and

3) $\lambda = 0$, that is $dS(RG) = 1$, if $G_p^{p^\alpha} = 1$ and either $G^{p^\alpha} = 1$ or $R^{p^\alpha}(p) = 0$.

Let $R_i, i \in I$, be a system of rings and let G be an arbitrary group. If $a \in (\Pi_{i\in I} R_i)G$, then

$$a = \sum_{g \in G_a} a_g g, \quad a_g \in \Pi_{i \in I} R_i,$$

where G_a is a finite subset of G. Mollov and Nachev (2010) note that G_a and the system $\{a_g | g \in G_a\}$ are defined identically from the element a. Besides, $a_g = (..., a_{gi}, ...)$ where $a_{gi} \in R_i$ for every $i \in I$ and every $g \in G_a$. They define a map

$$\varphi : (\Pi_{i\in I} R_i)G \to \Pi_{i\in I}(R_i G) \qquad (2.1)$$

by

$$\varphi(a) = (..., \sum_{g \in G_a} a_{gi} g, ...). \quad (2.2)$$

It is not hard to see that φ is a natural injective homomorphism of R-algebras.

Proposition 2.3. (Mollov and Nachev (2010a)) *The homomorphism (2.1) of R-algebras, defined by (2.2), is an isomorphism of R-algebras if and only if either I is a finite set or G is a finite group.*

Proposition 2.4. (Kuneva, Mollov, Nachev (2009)) *"Let G be an abelian group and let R be a finite commutative ring with identity of prime characteristic p without nilpotent elements. If α is any ordinal and G^{p^α} is finite, then*

$$f_\alpha(S/G_p)) = f_\alpha(S) - f_\alpha(G_p),$$

$$f_\alpha(S) = (|G^{p^\alpha}| - 2|G^{p^{\alpha+1}}| + |G^{p^{\alpha+2}}|) log_p|R|. \text{"}$$

Theorem 2.5. (Danchev (2004, Theorem 6, *(ii)*) *"Suppose $1 \neq G$ is an abelian group and R is an unitary perfect commutative ring without nilpotent elements in prime characteristic p .Then ...*

(ii) If $|R| \geq \aleph_0$ or $|G^{p^\sigma}| \geq \aleph_0$ for some ordinal σ,

$$f_\sigma(S(RG)/G_p) = \begin{cases} max(|R|, |G^{p^\sigma}|) & when \ \ |G_p^{p^\sigma}| \neq 1 \quad and \quad G^{p^\sigma} \neq G^{p^{\sigma+1}}; \\ 0, & when \ \ G_p^{p^\sigma} = 1 \quad or \quad G^{p^\sigma} = G^{p^{\sigma+1}}.\text{"} \end{cases}$$

3. Main results

Theorem 3.1. *Let G be an abelian group and let R be a commutative ring with identity of prime characteristic p. Suppose that G is a direct factor of $V(RG)$. Then $V(RG)/G$ is a p-group if and only if G satisfies exactly one of the following four conditions (*):*

1) $G = G_p$;

2) $G \neq G_p, tG = G_p$ and the ring R is indecomposable;

3) $p = 3, R^* = \langle -1 \rangle \times R_3^*, G = A \times G_3, |A| = 2$ and

4) $p = 2, R^* = R_2^*, G = A \times G_2, |A| = 3$ *and the equation* $X^2 + XY + Y^2 = 1 + N(R)$ *in the quotient ring* $R/N(R)$ *has only the trivial solutions in* $R/N(R)$.

Proof. Necessity. Let $V(RG)/G$ is a p-group. Since G is a direct factor of $V(RG)$ then $V(RG) = G \times T$. Consequently, $T \cong V(RG)/G$ is a p-subgroup of $V(RG)$. Therefore, $T \subseteq S(RG)$ and $V(RG) = GT \subseteq GS(RG) \subseteq V(RG)$. Hence $V(RG) = GS(RG)$. By a result of Mollov and Nachev (2010b) exactly one of the conditions (*) holds.

Sufficiency. Let G satisfies exactly one of the conditions (*). Consequently, by a result of Mollov and Nachev (2010b), $V(RG) = GS(RG)$ is fulfilled and

$$V(RG)/G = GS(RG)/G \cong S(RG)/G_p,$$

that is $V(RG)/G$ is p-group. □

Corollary 3.2. *Let* R *be a commutative ring of prime characteristic* p *and* G *be a direct factor of* $V(RG)$. *Then* $V(RG)/G$ *is a* p*-group if and only if* $V(RG) = GS(RG)$.

The proof follows directly by Theorem 3.1 and by results of Mollov and Nachev (2010b). □

If exactly one of the conditions (*) holds then we shall say that the pair (R, G) satisfies exactly one of the conditions (*).

Proposition 3.3. *Let* G *be an abelian group and let* R *be a commutative perfect ring with identity of prime characteristic* p. *Then for the maximal divisible subgroup* dT *of* $T = S(RG)/G_p$ *the following holds: if* α *is the first ordinal such that* $G^{p^\alpha} = G^{p^{\alpha+1}}$, *then* $dT = S(RG^{p^\alpha})G_p/G_p$, $dT \cong S(RG^{p^\alpha})/G_p^{p^\alpha}$ *and*

$$dT \cong \coprod_{\lambda} Z(p^\infty),$$

where

1) $\lambda = max(|R|, |G^{p^\alpha}|)$, *if* $G_p^{p^\alpha} \neq 1$;

2) $\lambda = max(|R(p)|, |G^{p^\alpha}|)$, *if* $G_p^{p^\alpha} = 1$, $G^{p^\alpha} \neq 1$ *and* $R(p) \neq 0$ *and*

3) $\lambda = 0$, *that is* $dT = 1$, *if* $G_p^{p^\alpha} = 1$ *and either* $G^{p^\alpha} = 1$ *or* $R(p) = 0$.

Proof. Since, by Lemma 2.1, G_p is a nice subgroup of $S(RG)$, then, as it is in the article of Mollov and Nachev (2010a), we see that

$$dT = S(RG^{p^\alpha})G_p/G_p \cong S(RG^{p^\alpha})/G_p^{p^\alpha}.$$

Therefore,

$$dT \cong S(RG^{p^\alpha})/G_p^{p^\alpha}. \tag{3.1}$$

(i) if $G_p^{p^\alpha} \neq 1$ then, as it is in the paper of Mollov and Nachev (2010), we see that case 1) of the proposition holds.

(ii) Let $G_p^{p^\alpha} = 1$. Then (3.1) implies

$$dT \cong dS(RG^{p^\alpha})$$

and, by Theorem 2.2, we obtain cases 2) and 3) of the proposition. □

The following result gives a full description, up to isomorphism, of the maximal divisible subgroup $dV(RG)$ of $V(RG)$.

Theorem 3.4. *Let R be a direct product of m commutative perfect rings R_i of prime characteristic p, $m \in \mathbb{N}$ and G be an abelian p-group. Suppose that for every $i = 1, 2, ..., m$, G is a direct factor of $V(R_iG)$ and the pair (R_i, G) satisfies exactly one of the conditions (*). Then there exist p-subgroups T_i of $V(R_iG), i = 1, 2, ..., m$, such that*

$$V(RG) \cong \coprod_m G \times \coprod_{i=1}^m T_i, \quad T_i \cong S(R_iG)/G_p. \tag{3.2}$$

and

$$dV(RG) \cong \coprod_m dG \times \coprod_{i=1}^m dT_i. \tag{3.3}$$

Let α be the first ordinal with the property $G^{p^\alpha} = G^{p^{\alpha+1}}$. Then $dT_i = S(R_iG^{p^\alpha})G_p/G_p$ and $dT_i \cong S(R_iG^{p^\alpha})/G_p^{p^\alpha}$.

Besides

$$dT_i \cong \coprod_\lambda Z(p^\infty) \tag{3.4}$$

where

1) $\lambda = max(|R_i|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} \neq 1$.

2) $\lambda = max(|R_i(p)|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} = 1$, $G^{p^\alpha} \neq 1$ and $R_i(p) \neq 0$ and

3) $\lambda = 0$, that is $dT_i = 1$, if $G_p^{p^\alpha} = 1$ and either $G^{p^\alpha} = 1$ or $R_i(p) = 0$.

Proof. We obtain, by Proposition 2.3, $RG \cong \prod_{i=1}^{m} R_iG$. Therefore

$$V(RG) \cong \prod_{i=1}^{m} V(R_iG). \tag{3.5}$$

Since G is a direct factor of $V(R_iG)$, then

$$V(R_iG) = G \times T_i, \quad T_i \cong V(R_iG)/G. \tag{3.6}$$

Since, by (*), $V(R_iG) = GS(R_iG)$, then by (3.6),

$$T_i \cong V(R_iG)/G = GS(R_iG)/G \cong S(R_iG)/G_p,$$

so that T_i is a p-subgroup of $V(R_iG)$ and the second formula of (3.2) holds. Besides (3.5) and (3.6) imply the first formula of (3.2). Then (3.2) implies obviously (3.3). Further the proof follows from Proposition 3.3. □

Theorem 3.5. *Let R be a direct product of m commutative perfect rings R_i with identity of prime characteristic p without nilpotent elements, $m \in \mathbb{N}$ and G be an abelian p-group. Suppose that for every $i = 1, 2, ..., m$, G is a direct factor of $V(R_iG)$ and $V(R_iG)/G$ is a simply presented p-group. Then there exist simply presented p-subgroups T_i of $V(R_iG), i = 1, 2, ..., m$, $T_i \cong S(R_iG)/G_p$, such that (3.2) holds. Every group T_i is described, up to isomorphism, by its Ulm-Kaplansky invariants $f_\alpha(T_i)$ and by its maximal divisible subgroup dT_i. The invariants $f_\alpha(T_i)$ are the following:*

(a) if R_i and G^{p^α} are finite, then

$$f_\alpha(T_i) = (|G^{p^\alpha}| - 2|G^{p^{\alpha+1}}| + |G^{p^{\alpha+2}}|) log_p|R_i| - f_\alpha(G_p). \tag{3.7}$$

(b) If either $|R_i| \geq \aleph_0$ or $|G^{p^\alpha}| \geq \aleph_0$, then

$$f_\alpha(T_i) = \begin{cases} max(|R_i|, |G^{p^\alpha}|), & if \ |G_p^{p^\alpha}| \neq 1 \quad and \quad G^{p^\alpha} \neq G^{p^{\alpha+1}}; \\ 0, & if\ either\ G_p^{p^\alpha} = 1 \quad or \quad G^{p^\alpha} = G^{p^{\alpha+1}}. \end{cases} \tag{3.8}$$

For the maximal divisible subgroup dT_i of T_i the following assertions hold: if α is the first ordinal such that $G^{p^\alpha} = G^{p^{\alpha+1}}$, then $dT_i \cong S(R_iG^{p^\alpha})/G_p^{p^\alpha}$. Besides for dT_i formula (3.4) is valid where

1) $\lambda = max(|R_i|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} \neq 1$.

2) $\lambda = max(|R_i(p)|, |G^{p^\alpha}|)$, if $G_p^{p^\alpha} = 1$, $G^{p^\alpha} \neq 1$ and $R_i(p) \neq 0$ and

3) $\lambda = 0$, that is $dT_i = 1$, if $G_p^{p^\alpha} = 1$ and either $G^{p^\alpha} = 1$ or $R_i(p) = 0$.

Proof. The proof is analogously of the proof of Theorem 3.3 of the paper of Mollov and Nachev (2010a). Namely, we obtain, by Proposition 2.3, that $RG \cong \prod_{i=1}^{m}(R_iG)$ and

$$V(RG) \cong \prod_{i=1}^{m} V(R_iG).$$

Since $V(R_iG) = G \times T_i$ and T_i is a p-group, then, by Corollary 3.2, $V(R_iG) = GS(R_iG)$. Hence

$$T_i \cong V(R_iG)/G = S(R_iG)G/G \cong S(R_iG)/G_p,$$

so that (3.2) holds. Since T_i is simply presented, then it is described, up to isomorphism, by $f_\alpha(T_i)$ and dT_i. In case (a) of the theorem, the invariants $f_\alpha(T_i)$ are given by Proposition 2.4 and in case (b) – by Theorem 2.5. Consequently, for $f_\alpha(T_i)$ (3.7) and (3.8) holds. Sinse $T_i \cong S(R_iG)/G_p$, then Proposition 3.3 implies the indicated description of dT_i □

Remark 1. For the calculation of the Ulm-Kaplansky invariants of the group $S(FG)/G_p$ in the finite case, that is, in case (a) of Theorem 3.5, we use Proposition 2.4, that is a result of Kuneva V. N., Mollov T. Zh. and Nachev N. A. (2009) and we do not use the result of Danchev [2004a, Theorem 6, case *(i)*], since the last result is inexact and it is not completed (see Kuneva V. N., Mollov T. Zh. and Nachev N. A. (2009)).

Remark 2. Let G and R be as in Theorem 3.5. In the paper of Mollov T. Zh. and Nachev N. A. (2010a) the following is noted:

a) the description of $d(S(RG)/G_p)$ in the article of of Danchev (2004b) is not given, up to isomorphism, although the author asserts the contrary and

b) the structures of $S(RG)$ and $V(RG)$ in the papers of Danchev (2004a and 2004b) are not completely determined.

Therefore, the cases *a) and b)* of the Remark 2 imply that our Proposition 3.3 and Theorem 3.5 are no corollaries from the results of Danchev (2004a and 2004b).

Acknowledgments

Research was partially supported by the fund "NI" of Plovdiv University, Bulgaria, under Contract FMI No 43 (2009).

References

BERMAN S. D. (1967a) Group algebras of countable abelian p-groups (Russian), *Dokl. Akad. Nauk SSSR*, **175**, No 3, 514-516.

BERMAN S. D. (1967b) Group algebras of countable abelian p-groups (Russian), *Publ. Math. Debrecen*, **14**, 365-405.

DANCHEV P. V. (2004a) Ulm-Kaplansky invariants for $S(RG)/G_p$, *Bull. Inst. Math. Acad. Sinica*, **32**, No. 2, 133-144.

DANCHEV P. V. (2004b) Maximal divisible subgroups in modular group algebras of p-mixed and p-splitting Abelian groups, *Rad. Mat.*, **13**, No. 1, 23-32.

FUCHS L. (1970 and 1973) Infinite abelian groups, Vol. I and II, *New York, Academic Press.*

KUNEVA V. N. (2006) Maximal divisible subgroups in modular group algebras of p-mixed abelian groups, *Compt. rend. Acad. bulg. Sci.*. **59**, 899-902.

KUNEVA V. N., MOLLOV T. ZH., NACHEV N. A. (2009) Some notes of the unit groups of commutative group algebras, *Plovdiv University "Paissii Hilendarsky", Bulgaria, Scientific works*, **36**, No 3, Mathematics, 67-88.

MAY, W. (1988). Modular group algebras of simply presented abelian groups. *Proc. Amer. Math. Soc.*, **104** (No 2): 403-409.

MAY W. (2008) Units of modular p-mixed abelian group algebras. Göbel, Rüdiger (ed.) et al., *Models, modules and Abelian groups.* In memory of A. L. S. Corner. Berlin: Walter de Gruyter , 267-276.

MAY W., MOLLOV T. ZH., NACHEV N. A. (2010) Isomorphism of modular group algebras of p-mixed abelian groups, *Communications in Algebra*, **38**, (to appear).

MOLLOV T. ZH. (1977) Ulm invariants of the Sylow p-subgroups of the group algebras of the abelian groups over a field of characteristic p (Russian), *Sixth Congress of the Bulgarian Mathematicians, Varna , Reports Abstracts, Section A2,*, p.2.

MOLLOV T. ZH. (1981) Ulm invariants of the Sylow p-subgroups for the group algebras of the abelian groups over a field of characteristic p (Russian), *PLISKA, Stud. Math. Bulg.*, **2**, 77-82.

MOLLOV T. ZH., AND NACHEV, N. A. (2010a) Unit groups of commutative group algebras, *International Electronic Journal of Pure and Applied Mathematics* **1**, No. 2, 163-175, http://math.uctm.edu/iejpam/.

MOLLOV T. ZH., NACHEV N. A. (2010b) Group of normalized units of commutative modular group rings, *Annales des sciences mathematiques du Québec*, **33**, No 1, 30-44.

Nachev N. A. (1995) Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p, *Communications in Algebra*, **23**(7), 2469–2489.

Nachev N. A., Mollov T. Zh. (1980) Ulm-Kaplansky invariants of the normed unit groups of the modular group rings of a primary abelian group (Russian), *Serdica Math. J.*, **6**, 258–263.

Department of Algebra,
University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: kuneva.1977@abv.bg, mollov@uni-plovdiv.bg

Received 21 July 2010

ВЪРХУ МУЛТИПЛИКАТИВНИТЕ ГРУПИ НА КОМУТАТИВНИ МОДУЛЯРНИ ГРУПОВИ АЛГЕБРИ

Велика Н. Кунева, Тодор Ж. Моллов

Резюме. Нека G е абелева група и R е комутативен пръстен с единица и проста характеристика p. Да означим с RG груповата алгебра на G над R, с $V(RG)$– групата от нормираните единици в RG, с G_p – p-компопнентата на G, с tG –периодичата подгрупа на G и с R_p^* – p-компопнентата на мултипликативната група R^* на пръстена R. В тази работа доказваме, че ако G е директен множител на $V(RG)$, то $V(RG)/G$ е p-група тогава и само тогава, когато двойката (R, G) удовлетворява точно едно от следните условия (*):

1) $G = G_p$;

2) $G \neq G_p, tG = G_p$ и пръстенът R е неразложим;

3) $p = 3, R^* = \langle -1 \rangle \times R_3^*, G = A \times G_3, |A| = 2$ и

4) $p = 2, R^* = R_2^*, G = A \times G_2, |A| = 3$ и уравнението $X^2 + XY + Y^2 = 1 + N(R)$ има само тривиалните решения във фактор-пръстена $R/N(R)$, където $N(R)$ е нил-радикалът на R.

Ако R е директно произедение на m комутативни перфектни пръстени R_i и G е директен множител на $V(R_iG)$, i=1,2,...m, то даваме пълно описание, с точност до изоморфизъм,

(i) на максималната делима подгрупа на $V(RG)$, ако всяка двойка (R_i, G), i=1,2,...m, удовлетворява точно едно от условията (*) и

(ii) на $V(RG)$, ако $V(R_iG)/G$, i=1,2,...m, са просто представени p-групи и пръстенът R е без нилпотентни елементи.