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ON SOME RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. A four-parametric family of 4-dimensional Riemannian product manifolds is constructed on a Lie group. This family is characterized geometrically. The form of the curvature tensor on the manifolds is obtained.

Key words: Riemannian almost product manifold, Riemannian metric, product structure, Lie group, Lie algebra

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1. Preliminaries

Let (M, P, g) be a $2n$-dimensional Riemannian almost product manifold, i.e. P is an almost product structure and g is a metric on M such that

$$P^2X = X, \qquad g(PX, PY) = g(X, Y) \tag{1.1}$$

for all differentiable vector fields $X, Y \in \mathfrak{X}(M)$.

Further, X, Y, Z, W (x, y, z, w, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

Let ∇ be the Levi-Civita connection of the metric g. Then, the tensor field F of type $(0, 3)$ on M is defined by

$$F(X, Y, Z) = g\left((\nabla_X P)Y, Z\right) . \tag{1.2}$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = -F(X, PY, PZ). \tag{1.3}$$

Let $\{e_i\}$ $(i = 1, 2, \dots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$. The Lie form α associated with F is defined by

$$\alpha(z) = g^{ij} F(e_i, e_j, z) \ . \tag{1.4}$$

The Nijenhuis tensor field N of the manifold is given by

$$N(X, Y) = [PX, PY] + [X, Y] - P[PX, Y] - P[X, PY]. \tag{1.5}$$

It is known [4] that the almost product structure P is product if and only if $N = 0$.

A classification of the Riemannian almost product manifolds is introduced in [4], where six classes of these manifolds are characterized according to the properties of F. The most general class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$ of Riemannian product manifold with $\operatorname{tr} P = 0$ is characterized by the condition [5]:

$$\begin{array}{c} \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6 : N(X, Y) = 0 \Leftrightarrow \\ F(X, Y, PZ) + F(Y, Z, PX) + F(Z, X, PY) = 0. \end{array} \tag{1.6}$$

Let R be the curvature tensor of ∇, i.e. $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. The corresponding tensor of type $(0, 4)$ is denoted by the same letter and it is given by $R(X, Y, Z, W) = g\,(R(X, Y)Z, W)\,.$

The Ricci tensor ρ and the scalar curvatures τ and $\overset{*}{\tau}$ of R are defined by:

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j), \quad \overset{*}{\tau} = g^{ij} \rho(e_i, Pe_j). \tag{1.7}$$

Definition 1.1. A tensor L of type $(0, 4)$ is called *a curvature-like tensor* if it satisfies the following conditions for any $X, Y, Z, W \in \mathfrak{X}(M)$:

$$\begin{array}{l} L(X, Y, Z, W) = -L(Y, X, Z, W) = -L(X, Y, W, Z); \\ L(X, Y, Z, W) + L(Y, Z, X, W) + L(Z, X, Y, W) = 0. \end{array} \tag{1.8}$$

Definition 1.2. [5] A curvature-like tensor L is called *a Kähler tensor* if it satisfies the following condition:

$$L(X, Y, PZ, PW) = L(X, Y, Z, W), \quad X, Y, Z, W \in \mathfrak{X}(M). \tag{1.9}$$

Further, we consider $2n$-dimensional Riemannian product manifolds with $\operatorname{tr} P = 0$.

1.1. Geometric properties of Riemannian product manifolds

It is known that the tensor P of type $(1,1)$ satisfies the identity:

$$(\nabla_X \nabla_Y P)\, Z - (\nabla_Y \nabla_X P)\, Z = R(X,Y)PZ - PR(X,Y)Z. \tag{1.10}$$

Take into acount (1.2), (1.3), $R(X,Y,Z,W) = g\,(R(X,Y)Z,W)$ and (1.1) we receive:

$$\begin{aligned}(\nabla_X F)\,(Y,Z,PW) - (\nabla_Y F)\,(X,Z,PW) = \\ = R(X,Y,PZ,PW) - R(X,Y,Z,W)\ ,\end{aligned} \tag{1.11}$$

$$\begin{aligned}(\nabla_X F)\,(Y,PZ,W) = -\,(\nabla_X F)\,(Y,Z,PW) - \\ -\,g\,((\nabla_X P)Z,(\nabla_Y P)W) - g\,((\nabla_X P)W,(\nabla_Y P)Z)\,.\end{aligned} \tag{1.12}$$

Theorem 1.1. *Let (M,P,g) be a Riemannian product manifold. Then, the curvature tensor R satisfies:*

$$\begin{aligned}&\underset{X,Y,Z}{\mathfrak{S}}\{R(PX,PY,Z,W) + R(X,Y,PZ,PW)\} + \\ &\underset{X,Y,Z}{\mathfrak{S}} g(\nabla_X P)Y - (\nabla_Y P)X, (\nabla_Z P)W - (\nabla_W P)Z = 0\ ,\end{aligned} \tag{1.13}$$

where $\mathfrak{S}$ is the cyclic sum by three arguments.

Proof. Since (M,P,g) belongs to the class $W_2 \oplus W_3 \oplus W_5 \oplus W_6$ then the characteristic condition (1.6) holds. By covariant differentiation in (1.6) we obtain

$$\begin{aligned}&(\nabla_X F)\,(Y,Z,PW) + (\nabla_X F)\,(Z,W,PY) + \\ &\quad +(\nabla_X F)\,(W,Y,PZ) + g\,((\nabla_X P)W,(\nabla_Y P)Z) + \\ &\quad +g\,((\nabla_X P)Y,(\nabla_Z P)W) + g\,((\nabla_X P)Z,(\nabla_W P)Y) = 0.\end{aligned} \tag{1.14}$$

Taking into account the equalities (1.10), (1.11), (1.14) and after straightforward calculation we get (1.13). □

Definition 1.3. A curvature-like tensor L on a Riemannian product manifold with $\operatorname{tr} P = 0$ is said to be *anti-Kähler* if it has the property:

$$L(X,Y,PZ,PW) = -L(X,Y,Z,W), \quad X,Y,Z,W \in \mathfrak{X}(M). \tag{1.15}$$

Next, Theorem 1.1 and Definition 1.3 imply:

Corollary 1.1. *Let* (M, P, g) *be a Riemannian product manifold with* $\operatorname{tr} P = 0$ *and let* R *be an anti-Kähler tensor. Then, we have:*

$$\mathop{\mathfrak{S}}_{X,Y,Z} g\left((\nabla_X P)Y - (\nabla_Y P)X, (\nabla_Z P)W - (\nabla_W P)Z\right) = 0. \tag{1.16}$$

Further, let us denote:

$$K(X, Y, Z, W) = g\left((\nabla_X P)Y - (\nabla_Y P)X, (\nabla_Z P)W - (\nabla_W P)Z\right). \tag{1.17}$$

Then, because of (1.17) the tensor K has the properties:

$$K(X, Y, Z, W) = -K(Y, X, Z, W) = -K(X, Y, W, Z). \tag{1.18}$$

By (1.18), Corollary 1.1, and Definition 1.1 we establish that K is a curvature-like tensor on any Riemannian product manifold if the curvature tensor R is an anti-Kähler tensor. Moreover, by (1.5) and $N = 0$, it is easy to prove that

$$K(X, Y, PZ, PW) = -K(X, Y, Z, W), \tag{1.19}$$

i.e. the tensor K is an anti-Kähler tensor, too.

2. A Lie group as a 4-dimensional Riemannian product manifold with $\operatorname{tr} P = 0$

Let V be a 4-dimensional real vector space and consider the structure of the Lie algebra defined by the brackets $[E_i, E_j] = C_{ij}^k E_k$, where $\{E_1, E_2, E_3, E_4\}$ is a basis of V and $C_{ij}^k \in \Re$. Then, the Jacobi identity for C_{ij}^k

$$C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0 \tag{2.1}$$

holds. Let G be the associated real connected Lie group and $\{X_1, X_2, X_3, X_4\}$ be a global basis of left invariant vector fields induced by the basis of V. We define an almost product structure on G by the conditions

$$PX_1 = X_3, \quad PX_2 = X_4, \quad PX_3 = X_1, \quad PX_4 = X_2. \tag{2.2}$$

Further, let us consider the left invariant metric defined by

$$g(X_i, X_i) = 1,\ i = 1, 2, 3, 4, \quad g(X_i, X_j) = 0 \text{ for } i \neq j. \tag{2.3}$$

Definition 2.1. [1] An almost product structure P on a Lie group G is said to be *Abelian* if

$$[PX, PY] = -[X, Y] \quad \text{for all } X, Y \in \mathfrak{g}. \tag{2.4}$$

The conditions (1.5) and (2.4) imply $N = 0$, i.e. P is a product structure. Thus, (G, P, g) is a Riemannian product manifold, i.e. $(G, P, g) \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$.

Proposition 2.1. *Let (G, P, g) be a 4-dimensional Riemannian product manifold and Abelian product structure P defined by (2.2). Then, the Lie algebra $\mathfrak{g}$ of G is given as follows:*

$$\begin{array}{ll} & [X_1, X_2] = -[X_3, X_4], \text{ i.e. } C_{12}^k = -C_{34}^k, \\ (2.5) & [X_1, X_4] = [X_2, X_3], \text{ i.e. } C_{14}^k = C_{23}^k, \\ & [X_1, X_3] = C_{13}^k X_k, \quad [X_2, X_4] = C_{24}^k X_k, \end{array}$$

where $C_{ij}^k \in \Re$ $(i, j, k = 1, 2, 3, 4)$ must satisfy the Jacobi identity.

Further, let us construct our example by setting

$$C_{12}^k = C_{34}^k = C_{14}^k = C_{23}^k = 0, \quad k = 1, 2, 3, 4.$$

In this case, for the non-zero Lie brackets of $\mathfrak{g}$ the Jacobi identity (2.1) implies

$$(2.6) \qquad [X_2, X_4] = aX_2 + bX_4, \quad [X_1, X_3] = cX_1 + dX_3,$$

where $a, b, c, d \in \Re$. Thus, the conditions (2.6) define a family of 4-dimensional real Lie algebras $\mathfrak{g}$, which is characterized by four parameters. It is known [1] that if a Lie algebra $\mathfrak{g}$ admits an Abelian product structure then $\mathfrak{g}$ is solvable. Therefore, the above considered Lie algebras (2.6) are solvable.

Let us remark that the Killing form [3] of the considered Lie algebra $\mathfrak{g}$

$$(2.7) \qquad B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y), \quad X, Y \in \mathfrak{g},$$

has the following form

$$B = \begin{pmatrix} d^2 & 0 & -cd & 0 \\ 0 & b^2 & 0 & -ab \\ -cd & 0 & c^2 & 0 \\ 0 & -ab & 0 & a^2 \end{pmatrix}.$$

It is easy to prove, that $\det B = 0$, i.e. the Killing form is degenerate. Thus, the Killing form B can not be a Riemannian metric.

2.1. Geometric characteristics of the constructed manifold

Let ∇ be the Levi-Civita connection of g. Then, the following well-known condition is valid

$$\begin{aligned} 2g(\nabla_X Y, Z) = & \, Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ & + g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X). \end{aligned} \tag{2.8}$$

Having in mind (2.3), (2.6) and (2.8), we obtain the following non-zero components of the Levi-Civita connection of the above constructed manifold (G, P, g):

$$\begin{array}{llll} \nabla_{X_1}X_1 = -cX_3, & \nabla_{X_3}X_1 = -dX_3, & \nabla_{X_4}X_2 = -bX_4, & \nabla_{X_2}X_2 = -aX_4, \\ \nabla_{X_1}X_3 = \ cX_1, & \nabla_{X_3}X_3 = \ dX_1, & \nabla_{X_4}X_4 = \ bX_2, & \nabla_{X_2}X_4 = \ aX_2. \end{array} \tag{2.9}$$

Then, by (2.2) and (2.9) for the non-zero components of ∇P we obtain:

$$\begin{array}{ll} (\nabla_{X_1}P)X_1 = \ \ 2cX_1, & (\nabla_{X_3}P)X_3 = \ \ 2dX_3, \\ (\nabla_{X_2}P)X_2 = \ \ 2aX_2, & (\nabla_{X_4}P)X_4 = \ \ 2bX_4, \\ (\nabla_{X_1}P)X_3 = -2cX_3, & (\nabla_{X_3}P)X_1 = -2dX_1, \\ (\nabla_{X_2}P)X_4 = -2aX_4, & (\nabla_{X_4}P)X_2 = -2bX_2. \end{array} \tag{2.10}$$

Next, taking into account (1.2), (1.4), (2.3) and (2.10), we get the non-zero components $F_{ijk} = F(X_i, X_j, X_k)$ of F and the components $\alpha_i = \alpha(Z_i)$ as follows:

$$\begin{array}{l} F_{111} = -F_{133} = 2c, \qquad F_{222} = -F_{244} = 2a, \\ F_{311} = -F_{333} = 2d, \qquad F_{422} = -F_{444} = 2b, \\ \alpha_1 = 2c, \quad \alpha_2 = 2a, \quad \alpha_3 = -2d, \quad \alpha_4 = -2b\,. \end{array} \tag{2.11}$$

2.2. Curvature properties of the constructed manifold

Let R be the curvature tensor of type $(0,4)$ of (G, P, g). Having in mind (2.9), we get the non-zero components $R_{ijks} = R(X_i, X_j, X_k, X_s)$ of R:

$$R_{1331} = -\left(c^2 + d^2\right), \qquad R_{2442} = -\left(a^2 + b^2\right). \tag{2.12}$$

Then, according to (2.2), (2.12) and Definition 1.3, we obtain:

Theorem 2.1. *The curvature tensor R of the manifold (G, P, g) is an anti-Kähler tensor and it has the form:*

$$R(X,Y,Z,W) = \frac{1}{4} g\left((\nabla_X P)Y - (\nabla_Y P)X, (\nabla_Z P)W - (\nabla_W P)Z\right). \tag{2.13}$$

Proof. Let $X = x^iX_i$, $Y = y^iX_i$, $Z = z^iX_i$, $W = w^iX_i$, where $x^i, y^i, z^i, w^i \in \mathrm{R}$ $(i = 1, 2, 3, 4)$, be arbitrary vectors in $\mathfrak{g}$. Then, by (2.12) for R we have

$$R(X,Y,Z,W) = \left(c^2+d^2\right)\left(x^1y^3-x^3y^1\right)\left(z^1w^3-z^3w^1\right) +(a^2+b^2)\left(x^2y^4-x^4y^2\right)\left(z^2w^4-z^4w^2\right). \tag{2.14}$$

Then, the equalities (1.16) and (2.10) imply that the right-hand side of (2.13) is equal to that of (2.14). □

Proposition 2.2. *The curvature tensor R of the manifold (G,P,g) satisfies the equation*

$$R(X,Y,Z,W) = g\left([X,Y],[Z,W]\right). \tag{2.15}$$

Proof. The validity of (2.15) follows from (2.6) and (2.14) by direct computation as in Theorem 2.1. □

Further, according to (2.9) and (2.12) we establish that

$$(\nabla_{X_i}R)(X_j,X_k,X_l,X_s) = 0 \text{ for all } i,j,k,l,s = 1,2,3,4 \tag{2.16}$$

and thus we obtain the following:

Proposition 2.3. *The manifold (G,P,g) is locally symmetric.*

Next, by vitrue of (1.7) and (2.12), we compute the non-zero components $\rho_{ij} = \rho(X_i,X_j)$ of the Ricci tensor and the value of scalar curvature τ as follows:

$$\rho_{11} = \rho_{33} = -(c^2+b^2), \quad \rho_{22} = \rho_{44} = -(a^2+b^2), \quad \tau = -2(a^2+b^2+c^2+d^2). \tag{2.17}$$

Therefore, by (2.2), we establish that ρ is a hibrid tensor with respect to P and the scalar curvature τ is constant. Further, according to (2.3) and (2.17), we prove the following:

Theorem 2.2. *The manifold (G,P,g) is Einsteinian if and only if $|a| = |c|$, $|b| = |d|$ hold.*

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ВЪРХУ НЯКОИ РИМАНОВИ МНОГООБРАЗИЯ СЪС СТРУКТУРА НА ПРОИЗВЕДЕНИЕ

Добринка Костадинова Щърбева

Резюме. Върху група на Ли е конструирано четири параметрично семейство от 4-мерни риманови многообразия със структура на произведение. Намерени са геометричните характеристики на това семейство от многообразия. Получен е видът на тензора на кривина за тези многообразия.