ПЛОВДИВСКИ УНИВЕРСИТЕТ "ПАИСИЙ ХИЛЕНДАРСКИ", БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 35, КН. 3, 2007 – МАТЕМАТИКА
PLOVDIV UNIVERSITY "PAISSII HILENDARSKI", BULGARIA
SCIENTIFIC WORKS, VOL. 35, BOOK 3, 2007 – MATHEMATICS

METRICS ON A MANIFOLD WITH A SEMI-TANGENTIAL STRUCTURE

A. H. Hristov

Abstract. Some properties of the partially projectable vector fields are proved and their help is used to determine the structure of the product P. There are specified requirements for the metric on the differentiation allowing metricizing of a manifold with a semi-tangential structure. It is proved that the three-dimensional manifold with a semi-tangential structure is locally conformal to an Euclidean one.

Key words: manifold with a semi-tangential structure, partially projectible matric, expanded lift of a metric

Mathematics Subject Classification 2000: 53C15

We suppose that E is a differentiable manifold of class C^∞, B is a submanifold of E of the same class ($\dim E = m+n,\ \dim B = n$), where $\sigma : E \to B$ is a submersion. Regarding σ, local triviality of E is required. If TB is the tangential differentiation of B then the point set

$$\mathfrak{M} = \{\sigma(p),\ \xi_{\sigma(p)}, p \mid p \in E,\ \xi_{\sigma(p)} - \text{a tangent vector to } B \text{ in point } \ \sigma(p)\}$$

is a manifold with a semi-tangential structure [1].

It is possible to introduce local coordinates (u^i, y^i, s^a) of point $\overset{*}{p} = (\sigma(p), \xi_{\sigma(p)}, p)$, $i = 1,2,\ldots,n$, $a = 1,2,\ldots,m$. The coordinates u^i determine the location of point $\sigma(p)$ from base B, while s^a determines the location of point p in layer $S\sigma(p) = \{q \mid q \in E, \sigma(q) = \sigma(p)\}$. $s^a = 0$ are the local equations of base B in E. If $\overset{*}{p}$ is in two coordinate surroundings simultaneously, then the relation between the corresponding coordinates is:

$$
\begin{aligned}
&\overline{u}^i = \varphi^i(\overline{u}^j) \\
(1) \qquad &y^i = \sum \frac{\partial \varphi^i(\overline{u}^j)}{\partial \overline{u}^k} . \overline{y}^k \\
&s^a = \theta^a(\overline{u}^k,\ \overline{s}^b)\ .
\end{aligned}
$$

The objective of the following publication is to demonstrate how $\mathfrak{M}$ can be metricized if E is in possession of the appropriate metrics.

Structure of the product P**.** The vector field on $E : x = x^i \frac{\partial}{\partial u^i} + x^a \frac{\partial}{\partial s^a}$ is called partially projectable if x^i are functions, depending only on the variables u^k. In particular $\frac{\partial}{\partial u^k}$ are exactly that kind of vector fields. In other words we have $\sigma_{*p}(x) = x^i(u^i)(\frac{\partial}{\partial u^i})_{\sigma(p)}$. There holds the following:

Proposition 1. *The partial projectability is retained at an arbitrary admissible change of the variables, which is determined by the first and the third equations of (1).*

Proof. The partial projectability of the vector field x means that the components of the matrix block X_1 in the matrix column $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, composed by the coordinates of the x field, depend only on the variables u^i. After the substitution in $E : (u^i, s^a) \to (\overline{u}^i, \overline{s}^a)$, according to the first and the third equations of (1), let the Jacobian be $\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$. In that case, with respect to the basis $\{\frac{\partial}{\partial \overline{u}^i}, \frac{\partial}{\partial \overline{s}^a}\}$ of the coordinates of x, the following matrix column has been defined

$$
\begin{pmatrix} \alpha^{-1}.X_1 \\ \cdots\cdots\cdots \\ -\gamma^{-1}\beta\alpha^{-1}X_1 + \gamma^{-1}X_2 \end{pmatrix} .
$$

That proves the proposition. □

If $x = x^i \frac{\partial}{\partial u^i}$ and $z = z^i \frac{\partial}{\partial u^i}$ are projectable, then the commutator $[x, z]$ is also a projectable vector field. Consequently, the totality of projectable vector fields on E, which will be called horizontal from now on, is Lie algebra. So in each point of E there has been defined the following horizontal distribution $H : \sigma_{*p}(H) = H_{\sigma(p)}$. Let V be $\operatorname{Ker} \sigma_*$.

The elements of $V : A = A^a(u_i, s_b)\frac{\partial}{\partial s^a}$ will be called vertical, and V - will be called a vertical distribution.

On the basis of the partial projectability definition of a vector field, there follows

Proposition 2. *1. If $x = x^i \frac{\partial}{\partial u^i} + x^a \frac{\partial}{\partial s^a}$ is partially projectable and $A = A^a \frac{\partial}{\partial s^a}$ is vertical, then the commutator $[x, A]$ is a vertical vector field.*
2. The distribution V is integrable. The integral surface in any point of E is a layer over the point $\sigma(p)$.

Proof. We have $X(A) = (X^i \frac{\partial}{\partial u^i} A^b + X^a \frac{\partial}{\partial s^a} A^b) \frac{\partial}{\partial s^b}$ and $A(X) = A^a \frac{\partial}{\partial s^a} X^b \frac{\partial}{\partial s^b}$. Consequently $[X, A] = X(A) - A(X)$ is a vertical field. The second part of the statement is proved in [1]. □

The availability of two distributions in E determines the structure of the product P. According to the basis $\{\frac{\partial}{\partial u^i}, \frac{\partial}{\partial s^a}\}$ we have the following form of $P : \begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix}$, where E_1 is the single matrix of degree n and E_2 is the single matrix of degree m. Then according to $\{\frac{\partial}{\partial \overline{u}^i}, \frac{\partial}{\partial \overline{s}^a}\}$ we obtain

$$P : \begin{pmatrix} E_1 & 0 \\ -2\gamma^{-1}\beta & -E_2 \end{pmatrix} .$$

Taking into consideration Propositions 1 and 2, there follows that P is integrable if and only if the horizontal distribution contains projectable vector fields only.

Metrics on $\mathfrak{M}$. In [2] we showed that if a partially projectable metric $g = (g_{\alpha\beta}),\ \alpha, \beta = 1, 2, \dots, n+m$ is assigned on E

$$g : \begin{bmatrix} g_1 & \vdots & A \\ \cdots & \cdot & \cdots \\ A^T & \vdots & B \end{bmatrix} ,$$

where $g_1 = (g_{ij}),\ A = (g_{ia}),\ B = (g_{ab}),\ i, j = 1, 2, \dots, n,\ a, b = n+1, \dots, n+m$, $\det g_1 . \det B \neq 0$, it can be expanded to the following g^{EC} in $\mathfrak{M}$

$$g^{EC} : \left[\begin{array}{c|ccc} \partial g_1 & g_1 & \vdots & A \\ \cdots & \cdots & \cdot & \cdots \\ g_1 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdot & \cdots \\ A^T & 0 & \vdots & B \end{array}\right] ,$$

0 is a zero matrix column, where $\partial g_1 = \sum y^i \partial_i g_1$.

If $A = 0$ then we will say that g^{EC} is in a canonical form. The metric g^{EC} is canonical if and only if the two distributions in E are orthogonal.

Theorem 1. *If h is a partially projectable metric on E, then it is possible to introduce a canonical metric on $\mathfrak{M}$.*

Proof. We use the fact that $h(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})_p = h(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})_{\sigma(p)}, P \circ \sigma_* = \sigma_* \circ P$ at any point p and we define $g : g(X, Y) = \frac{1}{2}[h(X, Y) + h(PX, PY)]$. In this case g induces metric g_1 on the horizontal distribution:

$$g_1\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}\right)_p = g\left(\sigma_* \frac{\partial}{\partial u^i}, \sigma_* \frac{\partial}{\partial u^k}\right)_{\sigma(p)} = h\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}\right)_{\sigma(p)}$$

By analogy g induces metric B on the vertical distribution

$$B\left(\frac{\partial}{\partial s^a}, \frac{\partial}{\partial s^b}\right) = h\left(\frac{\partial}{\partial s^a}, \frac{\partial}{\partial s^b}\right),$$

where $g_1\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial s^a}\right) = 0$ and $B\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial s^a}\right) = 0$. □

Theorem 2. *If the Riemannian metric g on E is partially projectable and the vertical distribution is one-dimensional, then it is possible to introduce local coordinates in the vicinity of a point, with respect to which the metric $\mathfrak{M}$ will be in a canonical form.*

Proof. A local basis in any point of the coordinate vicinity is $\{\frac{\partial}{\partial u^i}, \frac{\partial}{\partial s}\}$, $i = 1, 2, \ldots, n$. In that case, from the condition $\det(g_{\alpha\beta}) \neq 0$ follows $g_{n+1,n+1} = g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) \neq 0$. We substitute $(\overline{u}^i, \overline{s})$ for (u^i, s):

$$u^i = \overline{u}^i$$
$$s = \lambda(\overline{u}^k).\overline{s}, \qquad \lambda(\overline{u}^k) \neq 0 .$$

Respectively, the Jacobian is:

$$\begin{bmatrix} 1 & & & & \vdots & \\ & 1 & & & \vdots & \\ & & \ddots & & \vdots & \\ & & & 1 & \vdots & \\ \ldots & \ldots & \ldots & \ldots & \vdots & \ldots \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n & \vdots & \lambda \end{bmatrix},$$

where all the remaining elements equal zero, whereas $\lambda_i = \frac{\partial\lambda}{\partial u^i}$. The new local basis $\{\frac{\partial}{\partial \overline{u}^i}, \frac{\partial}{\partial \overline{s}}\}$ is represented by means of the old one in this way:

$$\frac{\partial}{\partial \overline{u}^i} = \frac{\partial}{\partial u^i} + \lambda_i \frac{\partial}{\partial s}$$
$$\frac{\partial}{\partial \overline{s}} = \lambda \frac{\partial}{\partial s} .$$

Consequently we have

$$\overline{g}_{i,n+1} = g(\frac{\partial}{\partial \overline{u}^i}, \frac{\partial}{\partial \overline{s}}) = \lambda[g_{i,n+1} + \lambda_i g_{n+1,n+1}] .$$

Subsequently, we apply the conditions

$$\overline{g}_{1,n+1} = \overline{g}_{2,n+2} = \ldots = \overline{g}_{n,n+1} = 0 .$$

This system is equivalent to

$$g_{i,n+1} + \lambda_i g_{n+i,n+i} = 0 . \tag{2}$$

From the first equation ($i = 1$) of the system (2) we find

$$\lambda(\overline{u}) = -\int \frac{g_{1,n+i}}{g_{n+1,n+1}} d\overline{u}^1 + C(\overline{u}^2, \ldots, \overline{u}^n) .$$

We determine the function $C(\overline{u}^2, \ldots, \overline{u}^n)$ from the second equation of (2)

$$g_{2,n+1} + \frac{\partial}{\partial \overline{u}^2}\left[-\int \frac{g_{1,n+1}}{g_{n+1,n+1}} d\overline{u}^1 + C(\overline{u}^2, \ldots, \overline{u}^n)\right] g_{n+1,n+1} = 0$$

accurated to a new function $D(\overline{u}^3, \ldots, \overline{u}^n)$. This process continues until we have exhausted all equations from (2), i.e. we have found $\lambda(\overline{u})$. □

Corollary. *If the Riemannian metric in the differentiation E is partially projectable, then the basis B, with respect to the Levi-Civita connection, generated by this metric, is a totally geodesic submanifold in E [3].*

As we have shown in [2], if g is a partially projectable metric on manifold E, then from the components of g there has been determined the matrix

$$g : \begin{bmatrix} g_1 & \vdots & A \\ \cdots & \cdot & \cdots \\ A^T & \vdots & B \end{bmatrix} ,$$

as the matrix blocks g_1 and B are symmetrical, and the elements of g_1 depend only on the variables $u^i (i = 1, 2, \dots, n)$. The matrix corresponds to the expanded lift g^{EC} of the metric g

$$g^{EC} : \begin{bmatrix} \partial g_1 & \vdots & g_1 & \vdots & A \\ \cdots & . & \cdots & . & \cdots \\ g_1 & \vdots & 0 & \vdots & 0 \\ \cdots & . & \cdots & . & \cdots \\ A^T & \vdots & 0 & \vdots & B \end{bmatrix} .$$

This matrix has degree $2n + m$ and if $(g^{EC}_{ij}) = \partial g_1$, then $\partial g_1 = (y^n \partial_{n+h} g_{ik}) = (y^n \frac{\partial}{\partial y^h} g_{ik})$. The components of the constant tensor field f from type (1,1) determine the matrix

$$f : \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & . & \cdots & . & \cdots \\ E & \vdots & & \vdots & \\ \cdots & . & \cdots & . & \cdots \\ & \vdots & & \vdots & \end{bmatrix} ,$$

where E is a one-dimensional block of degree n. The empty spaces mean zero matrix blocks. The metric g^{EC} is pure, i.e. $f^\sigma_\alpha \; g^{EC}_{\sigma\beta} = f^\sigma_\beta \; g^{EC}_{\alpha\sigma}$. The purity condition of g^{EC} proves to be very important. From that follows

Theorem 3. *If g is a partially projectable metric, then the partial derivatives of the components of g^{EC} are pure objects with relation to f:*

$$f^\varpi_\alpha \partial_\varpi g^{EC}_{\beta\gamma} = f^\varpi_\beta \partial_\alpha g^{EC}_{\varpi\gamma}, \;\; \alpha, \beta, \gamma = 1, 2, \dots, 2n{+}m \, . \tag{3}$$

Proof. We write the equations (3) in an equivalent form, taking into consideration the specific type of the matrix of f:

$$f^{n+i}_k = \delta^i_k, \;\; i, k = 1, 2, \dots, n \, .$$

This leads to the following equivalent notation of (3):

$$f^{n+i}_\alpha \partial_{n+i} g^{EC}_{\beta\gamma} = f^{n+i}_\beta \partial_\alpha g^{EC}_{n+i,\gamma} \, . \tag{4}$$

Subsequently we take into consideration that for any function Q

$$\partial_i Q = \frac{\partial Q}{\partial u^i}, \; \partial_{n+i} Q = \frac{\partial Q}{\partial y^i}, \; \partial_{2n+a} Q = \frac{\partial Q}{\partial s^a} \, .$$

We find that the components of the matrix blocks ∂g_1 and g_1 do not depend on the variables s^a, and those of A and B do not depend on y^i. It must be verified under these conditions that for all values of α, β, γ, obtained in the process of describing the sequence $1, 2, \ldots; n, n+1, \ldots, 2n; 2n+1, \ldots, 2n+m$, the equations in (4) are identically fulfilled. □

Corollary 1. *Since $f^{\alpha}_{\beta} = \mathrm{const}$, then for the linear connection ∇, the condition $\nabla f = 0$ is equivalent to purity of the connection coefficients in relation to f. In our case the purity of $\partial_\alpha g^{EC}_{\beta\gamma}$ leads to purity of the corresponding Christoffel symbols, generated by $g^{EC}_{\alpha\beta}$. Consequently, the semi-tangential tensor f is transferred in a parallel way in relation to the Levi-Civita connection, determined by g^{EC}.*

Corollary 2. *In any point of $\mathfrak{M}$ the pair $\{I, f\}$, where I is an identical transformation, determines a representation of algebra $\mathbb{R}(\varepsilon)$, $\varepsilon^2 = 0$. For this reason, the quadratic form with coefficients $g^{EC}_{\alpha\beta}$ is interpreted as a real model of such $\overset{*}{g}$ over $\mathbb{R}(\varepsilon)$ (see [1]). The coefficients of $\overset{*}{g}$ are analytical functions over $\mathbb{R}(\varepsilon)$. That is why we agree to call g^{EC} an analytical metric.*

The next example is an illustration of what has been stated up to this point. In it, the stratified manifold E is two-dimensional with a one-dimensional basis and layers. Let the linear element of E, corresponding to the metric of g is

$$d\tau^2 = du^2 + B(u)ds^2, \ B(u) > 0 .$$

We interpret E as a surface with a constant curvature in the three-dimensional Euclidean space. In the case of a sphere or a pseudosphere, let a fixed meridian (for example $s = 0$) be the basis B, and let the parallels be one-dimensional layers. When $B(u) = u^2$ we can consider E as a stratification of the circumferences

$$x^2 + y^2 = u^2$$

over $\mathbb{R}: \ s = 0, \ x = u\cos s, \ y = u \sin s$.

The metric g is partially projectable and because of the condition

$$\sigma_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}, \ \sigma_*\left(\frac{\partial}{\partial s}\right) = 0$$

there follows

$$g\left(\sigma_*\left(\frac{\partial}{\partial u}\right), \sigma_*\left(\frac{\partial}{\partial s}\right)\right) = 1$$

which is a projectable part. In that case the expanded metric g^{EC} over $\mathfrak{M}$ has a linear element

$$d\overset{*}{\tau}{}^2 = du.dy + B(u)ds^2 \ .$$

For this example, the semi-tangential structure over $\mathfrak{M}$ has a rank of 1.

For $\mathfrak{M}$ the only Christoffel symbols for g^{EC} are

$$\left\{ \begin{array}{c} 3 \\ 13 \end{array} \right\} = \frac{1}{2}\frac{B^{'}(u)}{B(u)} \quad \text{and} \quad \left\{ \begin{array}{c} 2 \\ 33 \end{array} \right\} = -\frac{1}{2}B^{'}(u).$$

It means that, with relation to the Levi-Civita connection for g^{EC} there follows

$$\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial s}\right) = \left\{ \begin{array}{c} 3 \\ 13 \end{array} \right\} \frac{\partial}{\partial s}, \ \nabla_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial s}\right) = \left\{ \begin{array}{c} 2 \\ 33 \end{array} \right\} \frac{\partial}{\partial y},$$

and $\frac{\partial}{\partial y}$ is an absolutely parallel vector field.

Let R be the corresponding Riemannian tensor curvature. The only nonzero component R is

$$R_{1331} = \frac{(B^{'})^2 - 2BB^{''}}{4B} \ .$$

The corresponding Ricci tensor S also has only one single component, distinct from zero:

$$S_{11} = \frac{1}{B}R_{1331} \ .$$

The scalar curvature at any point of $\mathfrak{M}$ is equal to zero.

Theorem 4. *The three-dimensional manifold $\mathfrak{M}$ is locally conformal of an Euclidean manifold.*

Proof. Let $v = \varphi(u) = \int \frac{1}{B(u)}du$ and $\psi(v)$ is the reverse function of $\varphi(u)$. Then we obtain

$$d\overset{*}{\tau}{}^2 = (dv.dy + ds^2).B(\psi(v)) \ .$$

In particular, if $B(u) = \cos^2 u$ we have

$$d\overset{*}{\tau}{}^2 = \cos^2 u \left[\frac{du.dy}{\cos^2 u} + ds^2\right] = \frac{1}{1+v^2}\left[dv.dy + ds^2\right] \ ,$$

$v = \operatorname{tg} u, \ \cos^2 u = \dfrac{1}{1+v^2}.$ □

References

[1] V.V. Vishnevskiy, On the geometric model of a semi-tangential structure. Higher School Bulletin, Mathematics, issue 3, 1983, pp.73–75.

[2] E.V. Pavlov, A.H. Hristov, Metric manifolds with a semi-tangential structure. Higher School Bulletin, Mathematics. (to appear)

[3] L.P. Eisenhart, Riemannian Geometry. Moscow, GIIL 1948.

Faculty of Mathematics and Informatics
University of Plovdiv
236 Bulgaria Blvd.,
4003 Plovdiv, BULGARIA
e-mail: hristov_asen@mail.bg

Received 28 April 2006

МЕТРИКИ ВЪРХУ МНОГООБРАЗИЯ С ПОЛУДОПИРАТЕЛНА СТРУКТУРА

А. Христов

Резюме. Доказани са някои свойства на частично проектируемите векторни полета и с тяхна помощ е определена структура на произведение P. Посочени са изисквания за метриката върху разслоението, позволяваща метризиране на многообразие с полудопирателна структура. Доказано е, че тримерното многообразие с полудопирателна структура е локално-конформно на евклидово.