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A DIRECTION IN THE METHOD OF MATRIX LYAPUNOV FUNCTIONS AND STABILITY IN TERMS OF TWO MEASURES

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Abstract. A new approach is applied in the method of matrix Lyapunov functions and the stability of systems of ordinary differential equations in terms of two measures is investigated.

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***Key words*:** (h_0, h)-stability, Lyapunov matrix-function, h-positive definite, h_0-decrescent, weakly h_0-decrescent.

1. Introduction

In the [5] and [8] the authors discuss stability properties in terms of two measures employing perturbing families of Lyapunov functions.

Analysing the stability, asymptotic stability and instability of systems of ordinary differential equations Martynyuk A.A. in [1] and [4] applies the method of matrix Lyapunov functions.

In the theory of the stability of large scale systems the different dynamic properties may have independent subsystems, but the whole system may possess certain type of stability on all variables. In the [2] and [11] the analysis of the polystability of dynamic systems is based on using matrix Lyapunov functions.

The extension of the method of matrix Lyapunov functions and the idea of the polystability [2], [11] and some ideas of the comparison method [6] allow a new approach to be outlined in investigating the stability of motion described in the paper [3].

Moreover the method of matrix Lyapunov functions and the idea of stability in terms of two measures are used in the mathematical models of the populations [9] and in the impulsive systems [10].

Using the ideas in [8] in this paper we investigate the stability in terms of two measures of a system of differential equations with the help of matrix Lyapunov functions applying a new approach in which the comparison system has a cascade structure.

2. Preliminary notes

Let $(R^n, \| \bullet \|)$ be a real Euclidean normed space, $R_+ = [0, +\infty),\ C[X, Y]$ - class of continuous mappings of the topological space X, in the topological space Y, $I = [\tau, +\infty),\ \tau \in R,\ I \in R$ - set of the initial values t_0.

We consider the system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_s),\ x_i(t_0) = x_{i0},\ i = 1, 2, \dots, s, \tag{1}$$

where $x_i \in R^{n_i},\ t \in I,\ f_i \in C[I \times R^{n_i} \times \cdots \times R^{n_s}, R^{n_i}]$ and we assume that $f_i(t, 0, \dots, 0) = 0,\ i = 1, 2, \dots, s$ for each $t \in I$.

Let us list the following classes of functions:

$K = \{\sigma \in C[R_+, R_+] : \sigma(u)$ is strictly increasing in u and $\sigma(0) = 0\}$;

$CK = \{\sigma \in C[R_+ \times R_+, R_+] : \sigma(t, u) \in K$ for each $t \in R_+\}$;

$\Gamma = \{h \in C[R_+ \times R^n, R_+] : \inf\limits_{x \in R^n} h(t, x) = 0$ for each $t \in R_+\}$.

In the problem for stability in terms of two measures the custom is system (1) to be considered in the region $S(h, \rho)$, where

$$S(h, \rho) = \{(t, x) \in R_+ \times R^n : h(t, x) < \rho\},\ \rho = \text{const} > 0. \tag{2}$$

We shall use the following definitions:

Definition 1 [5]. Let $h_0, h \in \Gamma$. Then we say that h_0 is finer than h if there exist a $\rho > 0$ and a function $\Phi \in CK$ such that $h_0(t, x) < \rho$ implies $h(t, x) \leq \Phi(t, h_0(t, x))$.

Definition 2 [5]. The system (1) is said to be (h_0, h)-equistable if given $\varepsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ that is continuous in t_0 for each ε such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$ for each $t \geq t_0$.

Definition 3 [6]. Let $Q \in C[R^s_+, R_+]$ with $Q(0) = 0$ and assume that $Q(u)$ is nondecreasing in u. Then we say that $Q \in K^*[R^s_+, R_+]$.

With the help of the matrix-function

$$U(t, x) = [u_{ij}(t, x)],\ i, j = 1, 2, \ldots, s \tag{3}$$

of the constant matrix A with $(s \times s)$-dimension and of the vector-function $\varphi \in C[R^n, R^s_+],\ \varphi(0) = 0$, we construct the vector-function [3]

$$L(t, x, \varphi) = AU(t, x)\varphi(x). \tag{4}$$

Let the function $L \in C[I \times R^n \times R^s_+, R^s]$ and satisfy the locally Lipschitz condition in x. We define the right upper Dini derivative [3] of the function (4):

$$D^+_{(1)}L(t, x, \varphi) = AD^+_{(1)}U(t, x).\varphi(x) + AU(t, x)D^+_{(1)}\varphi(x),$$

where

$$\begin{aligned} D^+_{(1)}U(t, x) &= \lim_{r \to 0^+} \sup[U(t + r, x + rf(t, x)) - U(t, x)]/r, \\ D^+_{(1)}\varphi(x) &= \lim_{r \to 0^+} \sup[\varphi(x + rf(t, x)) - \varphi(x)]/r \end{aligned}$$

for $(t, x) \in I \times R^n$.

We shall deduce the following definitions:

Definition 4. Let $U \in C[S(h, \rho), R^{s \times s}],\ h_0,\ h \in \Gamma$ and the function $Q \in K^*[R^s_+, R_+]$. Then the matrix-function $U(t, x)$ is said to be h-positive definite if there exist a $\rho > 0$ and a function $b \in K$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq Q(L(t, x, \varphi))$.

Definition 5. Let $L \in C[I \times R^n \times R^s_+, R^s],\ h_0,\ h \in \Gamma$ and the function $Q \in K^*[R^s_+, R_+]$. Then the vector-function $L(t, x, \varphi)$ is said to be:

1) h_0-decrescent if there exist a $\rho_0 > 0$ and a function $a_0 \in K$ such that $h_0(t, x) < \rho_0$ implies $Q(L(t, x, \varphi)) \leq a_0(h_0(t, x))$;

2) weakly h_0-decrescent if there exist a $\rho_0 > 0$ and a function $a \in CK$ such that $h_0(t, x) < \rho_0$ implies $Q(L(t, x, \varphi)) \leq a(t, h_0(t, x))$.

In order with system (1) and vector-function (4) we shall examine also the comparison system

$$\frac{du}{dt} = g(t,u), \; u(t_0) = u_0 \in R_+, \tag{5}$$

where $g \in C[I \times R^s_+, R^s], \; g(t,0) = 0$ for each $t \in I$.

Let $L = (L_p^T, L_q^T)^T$, where $L_p \in C[I \times R^n \times R^s_+, R^p], \; L_q \in C[I \times R^n \times R^s_+, R^q], \; p+q=s$.

Let $u(t;t_0,u_0)$ is a solution of system (5) with initial conditions $t_0 \in I$ and $u(t_0;t_0,u_0) = u_0 \in R_+$. We divide the vector $u \in R^s_+$ into two subvectors u_p and u_q such that $(u_p^T, u_q^T)^T = u$.

Definition 6 [6]. Let $Q_1 \in K^*[R^p_+, R_+]$ and $Q_2 \in K^*[R^q_+, R_+]$ $(p+q=s)$. Then system (5) is said to be polystable in I if for given $\varepsilon_1 > 0, \; \varepsilon_2 > 0$ and $t_0 \in R_+$ there exist a $\delta_1 = \delta_1(t_0,\varepsilon_1) > 0$ and a $\delta_2 = \delta_2(\varepsilon_2) > 0$ such that

$$\begin{aligned} &Q_1(u_{op}) < \delta_1 \text{ implies } Q_1(u_p(t;t_0,u_0)) < \varepsilon_1, \; t \geq t_0, \\ &Q_2(u_{oq}) < \delta_2 \text{ implies } Q_2(u_q(t;t_0,u_0)) < \varepsilon_2, \; t \geq t_0 \end{aligned} \tag{6}$$

3. Main result

Theorem. *Let the following hypotheses fulfill:*
(H_0) $h_0, \; h \in \Gamma$ *and* h_0 *is finer than* h;

(H_1) *there exist a matrix-function (3), a constant matrix* A *and a vector-function* $\varphi \in C[R^n, R^s_+], \; \varphi(0) = 0$ *such that the vector-function* $L(t,x,\varphi)$ *is locally Lipschitzian in* x *in the region* $S(h,\rho)$ *from (2),* $L_p(t,x,\varphi)$ *is weakly* h_0*-decrescent and*

$$b(h(t,x)) \leq Q_2(L_q(t,x,\varphi)) \leq a_0(h_0(t,x)) + a_1(Q_1(L_p(t,x,\varphi)))$$

for $(t,x) \in S(h,\rho) \cap S^c(h_0,\eta)$, *for every* $\eta (0<\eta<\rho)$ *and* $Q_1(L_p(t,0,\varphi(0))) \equiv 0$ *for every* $t \in I$, *where* $b, a_0, a_1 \in K[R_+, R_+], \; Q_1 \in K^*[R^p_+, R_+]$ *and* $Q_2 \in K^*[R^q_+, R_+]$ *with* $p+q=s$;

(H_2) *there exists a vector-function* $g \in C[I \times R^s_+, R^s]$, $g(t,u)$ *is quasi-monotone nondecreasing in* u, *for the components* $(g_p^T, g_q^T)^T = g$ *for which function the following inequalities are fulfilled:*

1) $D^+ L_p(t,x,\varphi) \le g_p(t, L_p(t,x,\varphi), 0)$ *for each* $(t,x) \in S(h,\rho)$;

2) $D^+ L_q(t,x,\varphi) \le g_q(t, L_p(t,x,\varphi), L_q(t,x,\varphi))$ *for each* $(t,x) \in S(h,\rho) \cap S^c(h_0,\eta)$, *for every* $\eta (0<\eta<\rho)$, *where* $S^c(h_0,\eta)$ *is the complement of* $S(h_0,\eta)$;

(H_3) *the comparison system (5) is polystable in* I *in the sense of definition 6.*

Then, the system (1) is (h_0,h)*-equistable.*

Proof. Since $L_p(t,x,\varphi)$ is weakly h_0-decrescent, there exist a $\rho_1 (0<\rho_1\le\rho)$ and a $\Phi_0 \in CK$ such that

$$Q_1(L_p(t,x,\varphi)) \le \Phi_0(t, h_0, (t,x)) \ \text{ if } \ h_0(t,x) < \rho_1. \tag{7}$$

Also, h_0 being finer than h implies that there exist a $\rho_0 (0 < \rho_0 \le \rho_1)$ and a $\Phi_1 \in CK$ such that

$$h(t,x) \le \Phi_1(t, h_0(t,x)) \ \text{ provided } \ h_0(t,x) < \rho_0, \tag{8}$$

where ρ_0 is such that $\Phi_1(\rho_0) < \rho_1$.

Let $0<\varepsilon<\rho$ and $t_0 \in I$ be given. By hypothesis (H_3), given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $t_0 \in I$, there exist a $\delta_{10} = \delta_{10}(t_0,\varepsilon_1) > 0$ and a $\delta_{20} = \delta_{20}(\varepsilon_2) > 0$ such that

$$\begin{aligned} &Q_1(u_{op}) < \delta_{10} \ \text{ implies } \ Q_1(u_p(t;t_0,u_0)) < \varepsilon_1, \ t \ge t_0, \\ &Q_2(u_{oq}) < \delta_{20} \ \text{ implies } \ Q_2(u_q(t;t_0,u_0)) < \varepsilon_2, \ t \ge t_0 \end{aligned} \tag{9}$$

Since $a_0 \in K$ and $\Phi_1 \in CK$, we can find $\delta_1 = \delta_1(\varepsilon)$ such that

$$a_0(\delta_1) < \frac{1}{2}\delta_{20} \ \text{ and } \ \Phi_1(t_0,\delta_1) < \varepsilon. \tag{10}$$

Let $\varepsilon_2 = b(\varepsilon)$ and $\varepsilon_1 = a_1^{-1}(\frac{1}{2}\delta_{20})$. Choose $u_{op} = L_p(t_0, x_0, \varphi(x_0))$. Since $\Phi_0 \in CK$, $Q_1(L_p(t,0,\varphi(0))) \equiv 0$ and (7), it follows that there exists a $\delta_2 = \delta_2(t_0,\varepsilon) > 0$ such that, $\delta_2 \in (0, \min(\delta_1,\rho_1))$ and

$$\begin{aligned} &h_0(t_0,x_0) < \delta_2 \ \text{ implies} \\ &Q_1(L_p(t_0,x_0,\varphi(x_0))) \le \Phi_0(t_0, h_0(t_0,x_0)) < \delta_{10}. \end{aligned} \tag{11}$$

We set $\delta = \min(\delta_1, \delta_2)$ and suppose that $h_0(t_0, x_0) < \delta$. We note that because of (8) and (10), we have

$$h(t_0, x_0) \leq \Phi_1(t_0, h_0(t_0, x_0)) \leq \Phi_1(t_0, \delta) \leq \Phi_1(t_0, \delta_1) < \varepsilon. \tag{12}$$

We claim that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon,\ t \geq t_0$. If this is not true, because of (12) there exist a solution $x(t)$ of the system (1) with $h_0(t_0, x_0) < \delta$ and $t_2 > t_1 > t_0$ such that

$$\begin{gathered} h(t_2, x(t_2)) = \varepsilon < \rho,\ h_0(t_1, x(t_1)) = \delta_1(\varepsilon), \\ x(t) \in S(h, \varepsilon) \cap S^c(h_0, \eta) \ \text{ with } \ \eta = \delta_1(\varepsilon) \ \text{ for } \ t \in [t_1, t_2]. \end{gathered} \tag{13}$$

Then (H_2) implies

$$\begin{gathered} D^+ m_p(t) \leq g_p(t, m_p(t), 0),\ t_0 \leq t \leq t_2; \\ D^+ m_q(t) \leq g_q(t, m(t)),\ t_1 \leq t \leq t_2, \end{gathered} \tag{14}$$

where $m(t) = L(t, x(t), \varphi(x(t)))$. Hence by the comparison theorem [7] we have

$$\begin{gathered} m_p(t) \leq u_p(t; t_1, m(t_1)),\ t_1 \leq t \leq t_2; \\ m_q(t) \leq u_q(t; t_1, m(t_1)),\ t_1 \leq t \leq t_2 \end{gathered} \tag{15}$$

Let $u^*(t) = u(t; t_1, m(t_1))$ be the extension of $u(t)$ to the left of t_1 up to t_0 and $u^*(t_0) = u_0^*$. Choose $u_p(t_0) = L(t_0, x_0, \varphi(x_0))$ and $u_q(t_0) = u_{oq}^*$. Consider now the differential inequality

$$D^+ m_p(t) \leq g_p(t, m_p(t), u_q^*(t)),\ m_p(t_0) = u_p(t_0) \tag{16}$$

which by comparison theorem [7] yields

$$m_p(t) \leq u_p(t; t_0, u_0),\ t_0 \leq t \leq t_1,\ u_0 = (u_p^T(t_0), u_{oq}^{*T})^T. \tag{17}$$

Then it is clear that $u(t) = (u_p^T(t; t_0, u_0),\ u_q^{*T}(t; t_1, m(t_1)))$ is a solution of the system (5) on $[t_0, t_1]$. Using (13), (15) and (H_1), we obtain

$$\begin{aligned} b(\varepsilon) = b(h(t_2, x(t_2))) \leq Q_2(L_q(t_2, x(t_2), \varphi(x(t_2))) \leq \\ \leq Q_2(u_q(t_2; t_1, m(t_1))). \end{aligned} \tag{18}$$

But from (9) and (17), provided $Q_1(u_{op}) < \delta_{10}$ we get

$$Q_1(L_p(t_1, x(t_1), \varphi(x(t_1))) \leq Q_1(u_p(t_1; t_0, u_0)) < b_1^{-1}(\frac{1}{2}\delta_{20}(\varepsilon)).$$

From (H_1), (10) and (13) now we have

$$\begin{aligned} Q_2(L_q(t_1, x(t_1), \varphi(x(t_1))) \leq \\ \leq a_0(h_0(t_1, x(t_1))) + a_1(Q_1(L_p(t_1, x(t_1), \varphi(x(t_1))))) \leq \\ \leq a_0(\delta_1(\varepsilon)) + a_1(a_1^{-1}(\frac{1}{2}\delta_{20})) < \frac{1}{2}\delta_{20} + \frac{1}{2}\delta_{20} = \delta_{20} \end{aligned}$$

and therefore from (9) we get

$$Q_2(u_q(t_2; t1, m(t1))) < b(\varepsilon)$$

which contradicts (18). Hence the proof is complete. □

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ЕДНО НАПРАВЛЕНИЕ В МЕТОДА НА МАТРИЧНИ ФУНКЦИИ НА ЛЯПУНОВ И УСТОЙЧИВОСТ ПО ОТНОШЕНИЕ НА ДВЕ МЕРКИ

И. К. Русинов

Резюме. Приложен е нов подход в метода на матричните функции на Ляпунов и е изследвана устойчивостта на системи от обикновени диференциални уравнения по отношение на две мерки.