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ANTISYMMETRICAL POLYNOMIALS

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Absract. In this paper we give a strict proof of the basic theorem for antisymmetrical polynomials over an arbitrary field P. Any antisymmetrical polynomial $g(x_1, \dots, x_n)$ over a field P of characteristic different form 2 is a product of the discriminant $\Delta(x_1, \dots, x_n)$ and a symmetrical polynomial $f(x_1, \dots, x_n)$ over P.

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The theory of the antisymmetrical polynomials although it is too interesting for the application usually does not study in the universities courses of the algebra. With a luck it will be apply for the solution of many problems of the algebra [1].

Before we begin to discuss the antisymmetrical polynomials, we will explain schematically some of the basic concepts in the theory of polnomials of n variables $x_1, x_2, \dots, x_n$ and we will give one generalization of Bezout's theorem for polynomials of one unknown.

Lemma 1. *If polynomials $f_i(x_1, \dots, x_n)$ over the field P, $i = 1, \dots, k$ are pairwise coprime and divide polynomial $g(x_1, \dots x_n)$, then their product divides $g(x_1, \dots x_n)$.*

Really, the proof of the lemma follows from the fact that $P[x_1, x_2, \dots, x_n]$ is a factorial ring ([2], p.208).

Lemma 2. *If $x_1, \dots, x_n$ are different variables then polynomials $x_i - x_j$, $j > i$ are pairwise coprime.*

Proof. The greatest common divisor $(x_i - x_j, x_k - x_l)$ of polynomials $x_i - x_j$ and $x_k - x_l$ is

$$(x_i - x_j, x_k - x_l) = 1, \quad \text{or} \quad x_i - x_j.$$

Assuming that the second case holds we have

$$x_k - x_l = c(x_i - x_j).$$

Therefore either (i) $x_k = cx_i$, or (ii) $x_k = -cx_j$.

(i) Let $x_k = cx_i$. Then $k = i$, $c = 1$ and $l = j$, i.e. $x_k - x_l = x_i - x_j$, which is a contradicction.

(ii) Let $x_k = -cx_j$. Then $k = j$ and $c = -1$. Therefore $x_l = x_i$, i.e. $l = i$ and for $x_k - x_l$ we have $k = j > i = l$, i.e. $k > l$ which is a contradiction. Hence $(x_i - x_j, x_k - x_l) = 1$. The lemma is proved. □

The next statement shows, that the theorem for a division of polynomials with a quotient and a remainder holds when the divident $f(x)$ and the divisor $x - c$ are polynomials over an arbitrary ring R with identity.

Theorem 1. *If $f(x)$ is a polynomial over a ring R with identity and $c \in R$, then there exists uniquely determinated polynomial $q(x)$ over R, such that*

$$f(x) = (x - c)q(x) + r, \quad r \in R \tag{1}$$

Proof. The existence and the uniqueness of the polynomial $q(x)$ and the element $r \in R$ is proved as in the theorem for a division of polynomials with a quotient and a remainder. Besides the existence of $q(x)$ and $r \in R$ may be also proved by Horner's scheme. □

The next theorem is a generalization of Bezout's theorem for polynomials over a numerical fields.

Theorem 2. *An element c of the ring R with identity is a root of the polynomial $f(x)$ over R if and only if when $x - c$ divides $f(x)$.*

For the proof, which is the same as Bezout's theorem, we use Theorem 1 ([3], p.34).

Definition 1. The polynomial $g(x_1, \dots x_n)$ over field P is called antisymmetrical if it changes only its sign by the change of the order of two arbitrary its variables, i.e.

$$g(x_1, \dots, x_j, \dots, x_i, \dots, x_n) = -g(x_1, \dots, x_i, \dots, x_j, \dots, x_n). \tag{2}$$

If the charactaristic of the field P is 2 then the antisymmetrical polynomial $g(x_1, \dots, x_n)$ over P is a symmetrical polynomial, because in the equality (2) it holds $-g(x_1, \dots, x_n) = g(x_1, \dots, x_n)$, i.e. (2) is a definition of a symmetrical polynomial $g(x_1, \dots, x_n)$. That is why later on we will consider antisymmetrical polynomials only over a field P of charactaristic different from 2.

A simple example for antisymmetrical polynomial of two variables x and y over P is $\Delta(x, y) = x - y$. An antisymmetrical polynomial of three variables x, y and z over P is the following

$$\Delta(x, y, z) = (x - y)(x - z)(y - z)$$

and of n variables:

$$\begin{aligned}\Delta(x_1, \dots, x_n) = \prod_{i<j}(x_i - x_j) = \quad & (x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_n) \quad \times \\ & \times(x_2 - x_3)\dots(x_2 - x_n) \quad \times \\ & \times(x_{n-1} - x_n).\end{aligned}$$

The polynomial $\Delta(x_1, \dots, x_n)$ is called discriminant of the variables $x_1, \dots, x_n$.

It is obviously that the square of every antisymmetrical polynomial is symmetrical polynomial.

If $f(x_1, \dots, x_n)$ is a symmetrical polynomial over a field P, then $f(x_1, \dots, x_n).\Delta(x_1, \dots, x_n)$ is obviously an antisymmetrical polynomial, because $f(x_1, \dots, x_n)$ does not change by the transposition of arbitrary unknowns and $\Delta(x_1, \dots, x_n)$ changes only its sign. Namely by this way we get every antisymmetrical polynomial, i.e. it holds the following theorem.

Theorem 3. *Any antisymmetrical polynomial $g(x_1, \dots, x_n)$ over a field P of charactaristic different from 2 is a product of the discriminant $\Delta(x_1, \dots, x_n)$ and a symmetric polynomial $f(x_1, \dots, x_n)$ over P i.e.*

$$g(x_1, \dots, x_n) = \Delta(x_1, \dots, x_n) f(x_1, \dots, x_n). \tag{3}$$

Proof. The polynomial $\varphi(x_i) = g(x_1, \dots, x_i, \dots, x_n)$, $i < n$ examined like a polynomial of x_i over a ring $A = P[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ have a root

$x_j \in A$ for every $j > i$. Indeed, by (2) we obtain

$$\varphi(x_j) = g(x_1, \ldots, x_j, \ldots, x_j, \ldots, x_n) = -g(x_1, \ldots, x_j, \ldots, x_j, \ldots, x_n).$$

Therefore $2g(x_1, \ldots, x_j, \ldots, x_j, \ldots, x_n) = 0$, i.e. $2\varphi(x_j) = 0$. Since the charactaristic of P is different from 2 then $\varphi(x_j) = 0$. Therefore, by Theorem 2 it holds $\varphi(x_i) = (x_i - x_j)\psi(x_i)$, where the coefficients of $\psi(x_i)$ are from the ring A, i.e. $x_i - x_j$ divides $g(x_1, \ldots, x_n)$ for every $j > i$. Since, by Lemma 2 the polynomials $x_i - x_j$, $j > i$ are pairwise coprime and they divide $g(x_1, \ldots, x_n)$, then by Lemma 1 their product divides $g(x_1, \ldots, x_n)$. Therefore, (3) is fulfilled. The polynomial

$$f(x_1, \ldots, x_n) = \frac{g(x_1, \ldots, x_n)}{\Delta(x_1, \ldots, x_n)} \tag{4}$$

is symmetrical, because by the transposition of two arbitrary unknowns x_i and x_j, the numerator and the determinator simultaneously change their signs, since they are antisymetrical polynomials.

The theorem is proved. □

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АНТИСИМЕТРИЧНИ ПОЛИНОМИ

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Резюме. В тази статия се дава строго доказателство на основната теорема за антисиметрични полиноми над произволно поле P. Всеки антисиметричен полином $g(x_1, \ldots, x_n)$ над поле P с характеристика различна от 2 е произведение на дискриминантата $\Delta(x_1, \ldots, x_n)$ и симетричен полином $f(x_1, \ldots, x_n)$ над P.