ПЛОВДИВСКИ УНИВЕРСИТЕТ “ПАИСИЙ ХИЛЕНДАРСКИ”, БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 34, КН. 3, 2004 – МАТЕМАТИКА
PLOVDIV UNIVERSITY “PAISSII HILENDARSKI”, BULGARIA
SCIENTIFIC WORKS, VOL. 34, BOOK 3, 2004 – MATHEMATICS

L-PARALLEL NETS IN AN N-DIMENSIONAL SPACE OF WEYL

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Abstract. Let in an n-dimensional space of Weyl W_n be given the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$, defined by the independent fields of directions $\underset{\alpha}{v}^i (\alpha = 1, 2, \ldots, n)$.

We note by L_α the lines with tangents fields of directions $\underset{1}{v}^k + \ldots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \ldots + \underset{n}{v}^k$. The net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ whose fields of directions $\underset{\alpha}{v}^i (\alpha = 1, 2, \ldots, n)$ are transformed in parallel along the lines L_α is called an L-parallel one.

A net, allowing conforming transformation into an L-parallel one is called a conforming L-parallel.

By means of prolonged covariant differentiation characteristics of L-parallel nets, there have been found conforming L-parallel nets and spaces of Weyl, containing such nets.

***Mathematics Subject Classifications* 2000:** 53A15, 53A60.

***Key words*:** prolonged covariant differentiate, derivative equations, conforming transformation, transformed in parallel, coordinate net , chebyshevian net.

1. Preliminaries

Let in an n-dimensional space of Weyl $W_n(g_{is}, \omega_k)$ with fundamental tensor g_{is} and additional vector ω_k be given n independent fields of directions

$\underset{\alpha}{v}^i(\alpha = 1, 2, \dots, n)$. The net defined by fields of directions $\underset{\alpha}{v}^i$ will be noted by $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$. The reciprocal co vectors $\overset{\alpha}{v}_i$ of $\underset{\alpha}{v}^i$ are defined by the equations:

$$(1) \qquad \underset{\alpha}{v}^i \overset{\alpha}{v}_k = \delta^i_k \Leftrightarrow \underset{\alpha}{v}^i \overset{\beta}{v}_i = \delta^\beta_\alpha$$

In [2] the following derivative equations are worked out:

$$(2) \qquad \dot{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}^i, \ \dot{\nabla}_k \overset{\alpha}{v}_i = \underset{\sigma}{\overset{\alpha}{T}}_k \overset{\sigma}{v}_i, \ \sigma = 1, 2, \dots, n,$$

where $\dot{\nabla}$ is the prolonged covariant derivative. Following [2], the fundamental tensor g_{is} and its reciprocal tensor $g^{is}(g_{is}g^{ik} = \delta^i_k)$ satisfy the equations

$$(3) \qquad \dot{\nabla}_k g_{is} = 0, \ \dot{\nabla}_k g^{is} = 0.$$

The reciprocally simple and differentiable correspondence between the points of two spaces, at which the fundamental tensors of these spaces coincide, is called conforming [1, p. 161]. The two spaces are called conforming. A well known property of the conforming spaces is that the angles between the corresponding directions are saved.

Let p is the conforming transformation of $W_n(g_{is}, \omega_k)$ into an $\overset{*}{W}_n(\overset{*}{g}_{is}, \overset{*}{\omega}_k)$ at which the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ is being transformed into $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \dots, \underset{n}{\overset{*}{v}}) \in \overset{*}{W}_n$. According to [1, p. 161] we have:

$$(4) \qquad \overset{*}{g}_{is} = g_{is}, \ \overset{*}{g}^{is} = g^{is}, \ \overset{*}{\overset{\alpha}{v}}_i = \overset{\alpha}{v}_i, \ \underset{\alpha}{\overset{*}{v}}^i = \underset{\alpha}{v}^i.$$

The vector

$$(5) \qquad p_i = \omega_i - \overset{*}{\omega}_i,$$

is called a vector of conforming transformation [1].

Let the derivative equations (2) in the space $\overset{*}{W}_n$ be:

$$(6) \qquad \dot{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{P}}_k \underset{\sigma}{v}^i, \ \dot{\nabla}_k \overset{\alpha}{v}_i = -\underset{\sigma}{\overset{\alpha}{P}}_k \overset{\sigma}{v}_i$$

In accordance with [3], between the coefficients of (2) and (6) the following equation is valid:

$$\overset{\sigma}{\underset{\alpha}{P}}_k = \overset{\sigma}{\underset{\alpha}{T}}_k + p_s \overset{\sigma}{v}_k \underset{\alpha}{v}^s - p_m \; g^{mi} \; \overset{\sigma}{v}_i \; g_{ks} \; \underset{\alpha}{v}^s. \tag{7}$$

2. L-parallel nets in W_n.

Let the lines, defined by the field of directions $\underset{\alpha}{v}^i (\alpha = 1, 2, \dots, n)$ be noted by $(\underset{\alpha}{v})$, and the lines, defined by the field of directions $\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k$ be marked with L_α.

Definition 1. The net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ will be called L-parallel, if the field of directions $\underset{\alpha}{v}^i$ is transformed in parallel along the lines defined by $L_\alpha (\alpha = 1, 2, \dots, n)$.

Theorem 1. *The net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ *is L-parallel if and only if the following equations hold:*

$$\overset{v}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) = 0, \; v \neq \alpha; \; \alpha, v = 1, 2, \dots, n. \tag{8}$$

Proof. Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ be an L-parallel one. The field $\underset{\alpha}{v}^i$ is transformed in parallel along the lines L_α if and only if the equation [2] is valid:

$$(\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \dot{\nabla}_k \underset{\alpha}{v}^i = \lambda \underset{\alpha}{v}^i, \tag{9}$$

λ is an arbitrary multiplier.

From (2) we have

$$\begin{aligned}
&(\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \dot{\nabla}_k \underset{\alpha}{v}^i = \\
&\quad \overset{1}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \underset{1}{v}^i + \dots + \overset{\alpha-1}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \\
&\qquad + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \underset{\alpha-1}{v}^i + \overset{\alpha}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \underset{\alpha}{v}^i + \\
&\qquad + \overset{\alpha+1}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \underset{\alpha+1}{v}^i + \dots \\
&\qquad + \overset{n}{\underset{\alpha}{T}}_k (\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) \underset{n}{v}^i.
\end{aligned}$$

From that, where according to (9) we obtain (8).

Conversely, let equations (8) hold for the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$. Taking into account (2), from the last equations we obtain (9). Consequently, $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ is L-parallel. □

The net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ is a chebyshevian one of the first kind if and only if, conditions [2] are satisfied:

$$\overset{\sigma}{\underset{\alpha}{T}}_k \underset{\beta}{v}^k = 0,\ \alpha \neq \beta;\ \alpha, \beta, \sigma = 1, 2, \ldots, n.$$

Consequently, any chebyshevian net of the first kind is at the same time an L-parallel net as well.

If with Γ^k_{is} we denote the coefficients of connection of the space $W_n(g_{is}, \omega_k)$, then the following theorem is valid.

Theorem 2. *The coordinate net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ *is an L-parallel one if and only if the coefficients of connection* Γ^k_{is} *satisfy the equations:*

$$\Gamma^v_{1\alpha} + \ldots + \Gamma^v_{\alpha-1,\alpha} + \Gamma^v_{\alpha+1,\alpha} + \ldots + \Gamma^v_{n\alpha} = 0,\ \alpha \neq v. \tag{10}$$

Proof. Let the L-parallel net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ be a coordinate one. From (1) and (2) we obtain

$$\overset{v}{v}_i \dot{\nabla}_k \underset{\alpha}{v}^i = \overset{v}{\underset{\alpha}{T}}_k \tag{11}$$

According to [2], we have

$$\dot{\nabla}_k \underset{\alpha}{v}^i = \nabla_k \underset{\alpha}{v}^i + \omega_k \underset{\alpha}{v}^i, \tag{12}$$

where ∇_k is the covariant derivative.
From (8), (9) and (12) for the L-parallel net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ we find:

$$\overset{v}{v}_i(\nabla_k \underset{\alpha}{v}^i + \omega_k \underset{\alpha}{v}^i)(\underset{1}{v}^k + \ldots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \ldots + \underset{n}{v}^k) = 0,\ v \neq \alpha. \tag{13}$$

Since the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ is coordinate, from (8) and equation $\nabla_k \underset{\alpha}{v}^i = \partial_k \underset{\alpha}{v}^i + \Gamma^i_{ks} \underset{\alpha}{v}^s$ follows:

$$\Gamma^v_{1\alpha} + \ldots + \Gamma^v_{\alpha-1,\alpha} + \Gamma^v_{\alpha+1,\alpha} + \ldots + \Gamma^v_{n\alpha} + \delta^v_\alpha(\omega_1 + \ldots + \omega_{\alpha-1} + \omega_{\alpha+1} + \ldots + \omega_n) = 0.$$

Taking into account that $\delta^v_\alpha = 0 (\alpha \neq v)$ from the last equation we obtain (10).

Conversely, let in the parameters of the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ the coefficients of connection Γ^k_{is} satisfy condition (10). From that where, taking into consideration equations (11), (12) and (13), equation (8) follows, i.e., the coordinate net is an L-parallel one. □

3. Conforming L-parallel nets in W_n.

Let the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ be transformed into an L-parallel one $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ by the conforming transformation.

If $\underset{\alpha\beta}{\omega}$ is the angle between the fields of directions $\underset{\alpha}{v}^i$ and $\underset{\beta}{v}^i$, then, following [3], we have

$$(14) \qquad g_{is} \underset{\alpha}{v}^i \underset{\beta}{v}^i = \cos \underset{\alpha\beta}{\omega}.$$

Definition 2. Nets, allowing conforming transformation into L-parallel nets, are called conforming L-parallel nets.

Theorem 3. *The orthogonal net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ is a conforming L-parallel net if and only if the equations below are satisfied:*

$$(15) \qquad (\overset{\sigma}{\underset{\alpha}{T}}_k - \overset{v}{\underset{\alpha}{T}}_k)(\underset{1}{v}^k + \ldots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \ldots + \underset{n}{v}^k) = 0, \ v \neq \alpha, \ \sigma \neq \alpha.$$

Proof. Let the orthogonal net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in W_n$ be transformed by conforming transformation into the L-parallel net $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \ldots, \underset{n}{\overset{*}{v}}) \in \overset{*}{W}_n$. In accordance with (6) and Theorem 1, the net $(\underset{1}{\overset{*}{v}}, \underset{2}{\overset{*}{v}}, \ldots, \underset{n}{\overset{*}{v}}) \in \overset{*}{W}_n$ is L-parallel if and only if the equations are satisfied:

$$(16) \qquad \overset{v}{\underset{\alpha}{P}}_k(\underset{1}{v}^k + \ldots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \ldots + \underset{n}{v}^k) = 0, \quad v \neq \alpha; \ v, \alpha = 1, 2, \ldots, n.$$

Taking into account (7), and from the last equations we find:

$$(17) \qquad (\overset{v}{\underset{\alpha}{T}}_k + p_s \underset{\alpha}{v}^s \overset{v}{v}_k - p_m g^{mi} \overset{v}{v}_i g_{ks} \underset{\alpha}{v}^s)(\underset{1}{v}^k + \ldots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \ldots + \underset{n}{v}^k) = 0,$$
$$v \neq \alpha; \ v, \alpha = 1, 2, \ldots, n.$$

Since the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ is orthogonal, it follows that

(18) $$g_{is}\underset{\alpha}{v}^i\underset{\beta}{v}^s = 0, \ \alpha \neq \beta.$$

Then (17) takes the form

(19) $$\overset{v}{\underset{\alpha}{T}}_k(\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k) + p_s\underset{\alpha}{v}^s = 0, \ v = 1, 2, \dots, n.$$

The system (19) has one and only solution of the vector of conforming transformation p_s if and only if condition (15) is satisfied. □

Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ be a conforming L-parallel net. We introduce the denotations

(20) $$\underset{\alpha}{Q} = \overset{v}{\underset{\alpha}{T}}_k(\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k), \ \alpha \neq v.$$

Hence (19) takes the form:

$$p_s\underset{\alpha}{v}^s = \underset{\alpha}{Q},$$

from where for the vector of conforming transformation we find

$$p_s = \underset{\alpha}{Q}\overset{\alpha}{v}_s.$$

Theorem 4. *The coordinate net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ *is a conforming L-parallel net if and only if the coefficients of connection* Γ^k_{is} *satisfy the equation*

(21) $$\Gamma^\sigma_{1\alpha} - \Gamma^v_{1\alpha} + \Gamma^\sigma_{2\alpha} - \Gamma^v_{2\alpha} + \dots + \Gamma^\sigma_{\alpha-1,\alpha} - \Gamma^v_{\alpha-1,\alpha} + \dots + \Gamma^\sigma_{\alpha+1,\alpha} - \Gamma^v_{\alpha+1,\alpha} + \dots + \Gamma^\sigma_{n\alpha} - \Gamma^v_{n\alpha} = 0, \quad \sigma \neq \alpha, \ v \neq \alpha.$$

Proof. Let the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ be a conforming L-parallel one. From (15), taking into account equations (11) and (12), we obtain (21).

Conversely, let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in W_n$ be a coordinate one and equation (21) is satisfied. From equations (11), (12) and (21) there it follows (15). Consequently, the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ is a conforming L-parallel net. □

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Received June 2003

L-ПАРАЛЕЛНИ МРЕЖИ В N-МЕРНО ПРОСТРАНСТВО НА ВАЙЛ

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Резюме. Нека в n-мерно вайлово пространство W_n е зададена мрежата $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$, определена от независимите полета от направления $\underset{\alpha}{v}^i(\alpha = 1,2,\dots,n)$.

Линиите на които допирателните полета от направления са $\underset{1}{v}^k + \dots + \underset{\alpha-1}{v}^k + \underset{\alpha+1}{v}^k + \dots + \underset{n}{v}^k$, означаваме с L_α. Мрежа $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v})$, на която полетата от направления $\underset{\alpha}{v}^i(\alpha = 1,2,\dots,n)$ се пренасят паралелно по линиите L_α, наричаме L-паралелна.

В работата с помощта на продълженото ковариантно диференциране са намерени характеристики на L-паралелни мрежи, на конформно L-паралелни мрежи и на вайлови пространства, които съдържат такива мрежи.