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A NOTE ON CERTAIN SUBCLASSES OF P-VALENT FUNCTIONS

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Abstract. The aim of this paper is to obtain coefficient estimates and distortion theorems for functions belonging to the class $\sum^s(p,\,\lambda)$ of meromorphically p-valent functions.

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1. Introduction

Let $\sum(p)$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_k z^k \qquad (p \in N = \{1,\,2,\dots\}) \tag{1.1}$$

which are regular and p-valent in the punctured disk

$$U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \backslash \{0\}.$$

Let $\sum(p,\,\lambda)$ denotes the class of functions $f(z) \in \sum(p)$ that satisfy the condition:

$$\left| \frac{z^{p+1}\,[f'(z) + \lambda z f''(z)] + p\,[1 - \lambda(p+1)]}{-z^{p+1}\,[f'(z) + \lambda z f''(z)] + p\,[1 - \lambda(p+1)]} \right| < 1, \tag{1.2}$$

$z \in U$, $0 \le \lambda < \dfrac{1}{p+1}$, $p \in N$.

We note that

$$\sum(1,\,\lambda)=\sum^{\lambda}(1,\,-1)\qquad\left(0\le\lambda<\frac{1}{2},z\in U\right),$$

Patel [1], $\sum(1,\,0)=\sum_c$ [2]. We note that every function belonging to the class $\sum_c$ is meromorphic close to convex in U^*.

Let $\sum^s(p)$ be the subclass of $\sum(p)$ consists of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}|a_k|z^k \tag{1.3}$$

that are regular and p-valent in U^*.

Let us write

$$\sum^s(p,\,\lambda)=\sum(p,\,\lambda)\cap\sum^s(p).$$

We note that $\sum^s(p,\,0)=H(p;\,1,\,-1)$ where $H(p;\,1,\,-1)$ denotes the class introduced and studied by Mogra [3].

The object of this paper is to present some properties of functions in the classes $\sum(p,,\lambda)$ and $\sum^s(p,\,\lambda)$.

2. Coefficient estimates

Theorem 2.1. *Let $f(z)$ defined by (1.3) be analytic and p-valent in U^*. Then $f(z)\in\sum(p,\,\lambda)$ if and only if*

$$\sum_{k=p}^{\infty}k[\lambda(k-1)+1]|a_k|\le p[1-(p+1)\lambda]. \tag{2.1}$$

Proof. First of all, suppose that the function $f(z)$ given by (1.3) is in the class $\sum^s(p,\,\lambda)$. Then

$$\left|\frac{z^{p+1}\left[f'(z)+\lambda zf''(z)\right]+p\left[1-\lambda(p+1)\right]}{-z^{p+1}\left[f'(z)+\lambda zf''(z)\right]+p\left[1-\lambda(p+1)\right]}\right|=$$

$$=\left|\frac{\displaystyle\sum_{k=p}^{\infty}k[\lambda(k-1)+1]|a_k|z^{k+p}}{2p[1-(p+1)\lambda]-\displaystyle\sum_{k=p}^{\infty}k[\lambda(k-1)+1]|a_k|z^{k+p}}\right|<1 \tag{2.2}$$

for all $z \in U$. Using the fact that $\Re z \leq |z|$ for all z, it follows that

$$\Re \left\{ \frac{\displaystyle\sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k| z^{k+p}}{2p[1-(p+1)\lambda] - \sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k| z^{k+p}} \right\} \leq 1, \tag{2.3}$$

$z \in U$. Now choose values of z on the real axis so that $z^{p+1}\left[f'(z) + \lambda z f''(z)\right]$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values we obtain:

$$\sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k| \leq 2p[1-(p+1)\lambda] - \sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k|$$

or

$$\sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k| \leq p[1-(p+1)\lambda].$$

Next, in order to prove the converse, we observe that

$$\left| \frac{z^{p+1}\left[f'(z) + \lambda z f''(z)\right] + p\left[1-\lambda(p+1)\right]}{-z^{p+1}\left[f'(z) + \lambda z f''(z)\right] + p\left[1-\lambda(p+1)\right]} \right| \leq$$

$$\leq \quad \left| \frac{\displaystyle\sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k|}{\displaystyle 2p[1-(p+1)\lambda] - \sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k|} \right|, \; z \in U$$

where we have made use of (2.2) and set $|z| = 1$. The last expression is bounded above by 1 provided

$$\sum_{k=p}^{\infty} k[\lambda(k-1)+1]|a_k| \leq p[1-(p+1)\lambda]$$

which is true by the hypothesis. Hence $f(z) \in \sum(p, \lambda)$.

Corollary 1. *If $f(z)$ defined by (1.3) is in the class $\sum^s(p,\lambda)$, then*

$$|a_k| \le \frac{p[1-(p+1)\lambda}{k[\lambda(k-1)+1]} \tag{2.4}$$

where $0 \le \lambda < \dfrac{1}{p+1}$, $p \in N$.

Equality holds for the functions of the form

$$f_k(z) = \frac{1}{z^p} + \frac{p[1-(p+1)\lambda}{k[\lambda(k-1)+1]} z^k, \qquad (k = p,\, p+1, \dots) \tag{2.5}$$

3. Some properties of the class $\sum^s(p,\lambda)$

Theorem 3.1. *Let* $0 \le \lambda_1 < \lambda_2 < \dfrac{1}{p+1}$. *Then* $\sum^s(p,\lambda_2) \subset \sum^s(p,\lambda_1)$.

Proof. Since $\dfrac{\lambda(k-1)+1}{1-(p+1)\lambda}$ is increasing function of λ $\left(0 < \lambda \le \dfrac{1}{p+1}\right)$, it follows from Theorem 2.1 that

$$\sum_{k=p}^{\infty} \frac{l\,[\lambda_1(k-1)+1]}{1-(p+1)\lambda_1}|a_k| < \sum_{k=p}^{\infty} \frac{k\,[\lambda_2(k-1)+1]}{[1-(p+1)\lambda_2]}|a_k| \le p$$

for $f(z) \in \sum^s(p,\lambda_2)$. Hence $f(z) \in \sum^s(p,\lambda_1)$.

Corollary 2.

$$\sum^S(p,\lambda) \subset H(p;\, 1, -1).$$

The proof is now immediate because $\lambda \ge 0$.

4. Distortion Theorems

Theorem 4.1. *If* $f(z) = \dfrac{1}{z^p} + \displaystyle\sum_{k=p}^{\infty} |a_k| z^k$ *is in the class* $\sum^s(p,\lambda)$, $0 \le \lambda < \dfrac{1}{p+1}$ *and* $p \in N$ *then for* $0 < |z| = r < 1$, *we have*

$$\frac{1}{r^p} - \frac{1-(p+1)\lambda}{\lambda(p-1)+1} r^p \le |f(z)| \le \frac{1}{r^p} + \frac{1-(p+1)\lambda}{\lambda(p-1)+1} r^p. \tag{4.1}$$

Proof. Suppose that $f(z)$ is in the class $\sum^s(p,\,\lambda)$. In view of Theorem 1.1 we have

$$p[\lambda(p-1)+1]\sum_{k=p}^{\infty}|a_k| \leq \sum_{k=p}^{\infty}k[\lambda(k-1)+1]|a_k| \leq p[1-(p+1)\lambda]$$

which yields

$$\sum_{k=p}^{\infty}|a_k| \leq \frac{1-(p+1)\lambda}{\lambda(p-1)+1}. \tag{4.2}$$

Consequently, we obtain

$$|f(z)| \leq \frac{1}{r^p} + \sum_{k=p}^{\infty}|a_k|r^k \leq \frac{1}{r^p} + r^p\sum_{k=p}^{\infty}|a_k| \leq \frac{1}{r^p} + \frac{1-(p+1)\lambda}{\lambda(p-1)+1}r^p$$

by (4.2). This gives the right hand inequality of (4.1). Also

$$|f(z)| \geq \frac{1}{r^p} - \sum_{k=p}^{\infty}|a_k|r^k \geq \frac{1}{r^p} - r^p\sum_{k=p}^{\infty}|a_k| \geq \frac{1}{r^p} - \frac{1-(p+1)\lambda}{\lambda(p-1)+1}r^p$$

which gives the left hand side of (4.1).

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БЕЛЕЖКА ВЪРХУ НЯКОИ ПОДКЛАСОВЕ P-ЛИСТНИ ФУНКЦИИ

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Резюме. Целта на статията е да се получат коефициентни оценки и оценки за ръста на функциите от класа $\sum(p, \lambda)$, състоящ се от мероморфни p-листни функции.