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A NOTE ON BEST PROXIMITY POINTS FOR P–SUMMING CYCLIC ORBITAL MEIR-KEELER CONTRACTIONS

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1. Introduction and Preliminaries

The classical Banach contraction theorem has a lot of applications [1]. One of the interesting generalizations of Banach contraction is the well known Meir - Keeler contraction theorem [12]. In [3], the following notions are introduced.If A and B are non empty subsets of a metric space (X, d), and if $T : A \cup B \to A \cup B$ is such that $T(A) \subseteq T(B)$ and $T(B) \subseteq T(A)$, then T is called a cyclic map. A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = \mathrm{dist}(A, B)$, where $\mathrm{dist}(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$. In this paper a best proximity point is obtained for a map called cyclic contraction. It is further generalized in [2] by introducing a map called cyclic Meir Keeler contraction.

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If (X,d) is a metric space, $A_1, A_2, ..., A_p$ $(p \geq 2)$ are non empty subsets of X and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$, then T is called a p–cyclic map if $T(A_i) \subseteq A_{i+1}$, where we use the notation $A_{p+1} = A_1$ [9]. Results about best proximity points are obtained for p–cyclic contractions in [9], which is a generalization of cyclic contractions introduced in [3]. The results of [2] were generalized in [7] by introducing a map called p–cyclic Meir- Keeler contractions. Further these results are generalized in a new direction in [8] by introducing a map called cyclic orbital Meir-Keeler contraction. Further development of the cyclic orbital type of maps was investigated in [4], [5], [6]. The introduction of orbital type of cyclic maps eases a lot the verifications of the conditions that ensure the existence and uniqueness of best proximity points. We will illustrate this in the final section with an example.

The conditions of the p-cyclic contractions [9], p-cyclic Meir-Keeler contractions [7], p-cyclic orbital Meir-Keeler contractions [10] and weak p–cyclic Kannan contractions [15] are such that the distances between the adjacent sets need to be equal. This condition on the sets is relaxed in the p–summing maps introduced in [14] and further developed in [18].

We will use the convention $A_{p+j} = A_j$ for $j = 1, 2, \dots p$. Let us denote by $P = \sum_{j=1}^{p} \operatorname{dist}(A_j, A_{j+1})$,

$$s_p(x_1, x_2, ..., x_p) = \sum_{j=1}^{p-1} d(x_j, x_{j+1}) + d(x_p, x_1),$$

where if $x_1 \in A_i$ then $x_{1+k} \in A_{i+k}$ for every $k = 1, 2, ..., p-1$. From the definition of s_p it is easy to see that for any $x_{n_j} \in A_{i+j-1}$, $j = 1, 2, \dots, p$ there holds the equality

$$\begin{aligned} s_p(x_{n_1}, x_{n_2}, \dots, x_{n_p}) &= s_p(x_{n_p}, x_{n_1}, x_{n_2}, \dots, x_{n_{p-1}}) \\ &= s_p(x_{n_{p-1}}, x_{n_p}, x_{n_1}, x_{n_2}, \dots, x_{n_{p-2}}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= s_p(x_{n_2}, x_{n_3}, \dots, x_{n_p}, x_1). \end{aligned} \tag{1}$$

Two conditions (P.1) and (P.2) were imposed on the investigated maps in [18].

Definition 1. ([18]) Let A_i, $i = 1, 2 \dots, p$ be subsets of a metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a cyclic map. The map T is called a p–summing cyclic orbital Meir–Keeler contraction if there exists $x \in A_1$ with the properties

$$\begin{array}{c}\text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that}\\ \text{if there holds the inequality}\\ s_p(T^{pn-1}x, y_1, y_2, \ldots, y_{p-1}) < P + \varepsilon + \delta\\ \text{for } n \in \mathbb{N} \text{ and } y_i \in A_i, i = 1, 2 \ldots, p-1,\\ \text{then there holds the inequality}\\ s_p(T^{pn}x, Ty_1, Ty_2, \ldots, Ty_{p-1}) < P + \varepsilon.\end{array} \tag{P.1}$$

and

$$\begin{array}{c}\text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that}\\ \text{if there holds the inequality}\\ s_p(T^{pn}x, y_2, y_3, \ldots, y_p) < P + \varepsilon + \delta\\ \text{for } n \in \mathbb{N} \text{ and } y_i \in A_i, i = 2, 3 \ldots, p,\\ \text{then there holds the inequality}\\ s_p(T^{pn+1}x, Ty_2, Ty_3, \ldots, Ty_p) < P + \varepsilon.\end{array} \tag{P.2}$$

We have weaken condition (P1) and removed condition (P2) in the main result for existence and uniqueness of best proximity points. This not only increase the set of the Meir-Keeler type maps that have best proximity points but help us to verify easier the sufficient condition and therefore we were able to present an example with integral operators.

Deep results, that characterize the Meir–Keeler maps are obtained by introducing the notion of L–functions [11].

Definition 2. ([11]) A function $\phi : [0, \infty) \to [0, \infty)$ is called an L - function if $\phi(0) = 0$, $\phi(s) > 0$ for $s \in (0, \infty)$ and for every $s \in (0, \infty)$ there exists a $\delta > 0$ such that $\phi(t) \le s$ for every $t \in [s, s + \delta]$.

Lim also gave a set of equivalent conditions for L - functions [11]. Suzuki generalize Lim's results [17]. We will need the following lemma for the proof of the main results.

Lemma 1. *([17]) Let Y be a non empty set and let $f, g : Y \to [0, \infty)$. Then the following are equivalent:*

(i) For each $\epsilon > 0$ there exists a $\delta > 0$ such that $f(x) < \epsilon + \delta \Rightarrow g(x) < \epsilon$.

(ii) There exists an L - function ϕ (which may chosen to be a non decreasing and continuous) such that $f(x) > 0 \Rightarrow g(x) < \phi(f(x))$, $x \in Y$ and $f(x) = 0 \Rightarrow g(x) = 0$, $x \in Y$.

The next two lemmas are crucial in the investigation of best proximity points in uniformly convex Banach spaces.

Lemma 2. *([3]) Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in A and $\{y_n\}_{n=1}^\infty$ be a sequence in B satisfying:*

(i) $\lim_{n\to\infty} \|z_n - y_n\| = \mathrm{dist}(A,B)$;

(ii) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$ there holds the inequality $\|x_m - y_n\| \leq \mathrm{dist}(A,B) + \varepsilon$.

Then for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, there holds the inequality $\|x_m - z_n\| \leq \varepsilon$.

Lemma 3. *([3]) Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in A and $\{y_n\}_{n=1}^\infty$ be a sequence in B satisfying:*

(i) $\lim_{n\to\infty} \|x_n - y_n\| = \mathrm{dist}(A,B)$;

(ii) $\lim_{n\to\infty} \|z_n - y_n\| = \mathrm{dist}(A,B)$.

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 4. *([18] Let A, B be closed subsets of a strictly convex Banach space $(X, \|\cdot\|)$, such that $\mathrm{dist}(A,B) > 0$ and let A be convex. If $x, z \in A$ and $y \in B$ be such that $\|x - y\| = \|z - y\| = \mathrm{dist}(A,B)$, then $x = z$.*

2. Main Result

We will start with some notations, which we will introduce just to fit some of the formulas in the text field.

Let (X,d) be a metric space and $(X, \|\cdot\|)$ be a Banach space. For a Banach space $(X, \|\cdot\|)$ we will always consider the metric d, endowed by the norm, i.e. $d(x,y) = \|x - y\|$. Let $A_1, A_2, ..., A_p$ be non empty subsets of a metric space (X,d). We will use the convention $A_{p+j} = A_j$ for $j = 1, 2, \dots p$. Just to fit some of the formulas in the text field let us denote $s_{p,n,i,k}(x,y) = s_p(T^{pn-i}x, T^{pn-i+1}x, \dots,$ $T^{pn-i+k-1}x, T^{k-i}y, T^{k-i+1}y, ..., T^{p-i-1}y)$ for x, y, which belong to one and the same set A_j, $j = 1, 2, \dots, p$ and $k = 1, 2, \dots, p-1$.

We will write explicitly $s_{p,n,i,k}(x,y)$ for $i = 0, 1$ and $k = 1, 2, 3$ just to make clearer the above notation.

$$\begin{aligned} s_{p,n,1,1}(x,y) &= \rho(T^{pn-1}x, y) + \rho(y, Ty) + \cdots + \rho(T^{p-3}y, T^{p-2}y) \\ &\quad + \rho(T^{p-2}y, T^{pn-1}x), \end{aligned}$$

$$\begin{aligned}
s_{p,n,0,1}(x,y) &= \rho(T^{pn}x,Ty)+\rho(Ty,T^2y)+\cdots \\
&\quad +\rho(T^{p-2}y,T^{p-1}y)+\rho(T^{p-1}y,T^{pn}x),
\end{aligned}$$

$$\begin{aligned}
s_{p,n,1,2}(x,y) &= \rho(T^{pn-1}x,T^{pn}x)+\rho(T^{pn}x,Ty)+\rho(Ty,T^2y) \\
&\quad +\cdots+\rho(T^{p-3}y,T^{p-2}y)+\rho(T^{p-2}y,T^{pn-1}x),
\end{aligned}$$

$$\begin{aligned}
s_{p,n,0,2}(x,y) &= \rho(T^{pn}x,T^{pn+1}x)+\rho(T^{pn+1}x,T^2y) \\
&\quad +\rho(T^2y,T^3y)+\cdots+\rho(T^{p-2}y,T^{p-1}y) \\
&\quad +\rho(T^{p-1}y,T^{pn}x),
\end{aligned}$$

$$\begin{aligned}
s_{p,n,1,3}(x,y) &= \rho(T^{pn-1}x,T^{pn}x)+\rho(T^{pn}x,T^{pn+1}x) \\
&\quad +\rho(T^{pn+1}x,T^2y)+\rho(T^2y,T^3y)+\cdots \\
&\quad +\rho(T^{p-3}y,T^{p-2}y)+\rho(T^{p-2}y,T^{pn-1}x),
\end{aligned}$$

$$\begin{aligned}
s_{p,n,0,3}(x,y) &= \rho(T^{pn}x,T^{pn+1}x)+\rho(T^{pn+1}x,T^{pn+2}x) \\
&\quad +\rho(T^{pn+2}x,T^3y)+\rho(T^3y,T^4y)+\cdots \\
&\quad +\rho(T^{p-2}y,T^{p-1}y)+\rho(T^{p-1}y,T^{pn}x),
\end{aligned}$$

Definition 3. Let $A_1, A_2, ..., A_p$ be non empty subsets of a metric space (X,d). Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map. The map T is called a p–summing cyclic orbital Meir-Keeler contraction of type 2 with a constant D if there exists $x \in A_1$ with the property for every $\varepsilon > 0$ there is $\delta > 0$, such that for all $k = 1, 2, \dots p-1$ and all $y \in A_1$ there holds

$$\begin{array}{c}
\text{if } s_{p,n,1,k}(x,y) < D+\varepsilon+\delta \text{ for } n \in \mathbb{N} \text{ and } y \in A_1 \\
\text{then there holds the inequality } s_{p,n,0,k}(x,y) < D+\varepsilon
\end{array} \tag{1}$$

We call these maps of type 2, because they are different from the maps introduced in [18].

Theorem 4. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a complete metric space (X,ρ). Let T be a p–summing cyclic orbital Meir-Keeler contraction of type 2 with a constant $D = 0$. Then there exists a unique $\xi \in \bigcap_{i=1}^p A_i$, such that:*

(a) $T\xi = \xi$;

(b) for any $x \in A_1$, that satisfies (1) there holds $\lim_{n\to\infty} T^{pn}x = \xi$.

Theorem 5. *Let $A_1, A_2, ..., A_p$ be non empty closed and convex subsets of a uniformly convex Banach space $(X, \|.\|)$. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p–summing cyclic orbital Meir-Keeler contraction of type 2 with a constant D equal to P or to zero. Then there exists a unique point, say $\xi \in A_1$, such that:*

(a) for every $x \in A_1$ satisfying (1)*, the sequence $\{T^{pn}x\}$ converges to ξ;*

(b) ξ *is a best proximity point of* T *in* A_1 *and* $T^j\xi$ *is a best proximity point of* T *in* A_{1+j}*;*

(c) ξ *is a fixed point for the map* $T^p : A_1 \to A_1$*.*

3. Auxiliary Results

Lemma 5. *Let* (X, d) *be a complete metric space. Let* A_i*,* $i = 1,...,p$ *be non empty subsets of* X*. Let* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a* p*–summing cyclic orbital Meir-Keeler contraction of type* 2 *with a constant* D *equal either to* P *or to zero. Then*

(i) the inequality $s_{p,n,0,k}(x, y) \leq s_{p,n,1,k}(x, y)$ *holds for* $x \in A_1$*, which satisfies (1),* $n \in \mathbb{N}$ *and* $y \in A_1$*;*

(ii) $s_p(T^j x, T^{j+1}x, \dots, T^{j+p-1}x) \leq s_p(T^{j-1}x, T^j x, \dots, T^{j+p-2}x)$*;*

(iii) $\lim_{n\to\infty} s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) = P$ *for* $x \in A_1$ *satisfying (1).*

Proof. I) Let us consider the case $D = P$.

(i) Let $x \in A_1$ satisfy (1). Define the following sets: $C_j = \{T^{pn-1+j}x : n \in \mathbb{N}\}$ for $j = 0, 1, \dots, p-2$ and $B_k = \{T^{k-1}y : y \in A_1\}$, for $k = 1, \dots, p-1$. Define

$$f_k, g_k : C_0 \times C_1 \times \cdots \times C_{k-1} \times B_k \times B_{k+1} \times \cdots \times B_{p-1} \to [0, \infty)$$

as follows: $f_k(x, y) = s_{p,n,1,k}(x, y) - P$ and $g_k(x, y) = s_{p,n,0,k}(x, y) - P$.

We will write explicitly f_k for $k = 1, 2, 3$ just to make clearer the above definition

$$f_1 : C_0 \times B_1 \times B_2 \times \cdots \times B_{p-1} \to [0, \infty),$$

$$f_2 : C_0 \times C_1 \times B_2 \times B_3 \times \cdots \times B_{p-1} \to [0, \infty),$$

$$f_3 : C_0 \times C_1 \times C_2 \times B_3 \times B_4 \times \cdots \times B_{p-1} \to [0, \infty).$$

Then f_k and g_k satisfy the condition (1) of Lemma 3. Hence there exists an L function ϕ_k such that

$$s_{p,n,0,k}(x, y) - P < \phi_k(s_{p,n,1,k}(x, y) - P), \text{ if } s_{p,n,1,k}(x, y) > P$$

and

$$s_{p,n,0,k}(x, y) - P = \phi_k(s_{p,n,1,k}(x, y) - P), \text{if} s_{p,n,1,k}(x, y) = P.$$

From the definition of the L function, it follows that

$$s_{p,n,0,k}(x,y) < s_{p,n,1,k}(x,y) \text{ when } s_{p,n,0,k}(x,y) > 0 \tag{2}$$

$$s_{p,n,0,k}(x,y) = s_{p,n,1,k}(x,y) \text{ when } s_{p,n,1,k}(x,y) = 0, \tag{3}$$

for all $n \in \mathbb{N}$, all $y \in A_i$ and all $k = 0, 1, 2, \dots p-1$. Therefore (i) is proved.

(ii). For any $j \in \mathbb{N}$ there exists $k = 0, 1, \dots, p-1$, so that $j + k = pn$. Then using (1) and (i) we get the chain of inequlities

$$\begin{array}{rcl} r_j & = & s_p(T^j x, T^{j+1}x, \dots, T^{j+k-1}x, T^{j+k}x, T^{j+k+1}x, \dots, T^{j+p-1}x) \\ & = & s_p(T^{j+k}x, T^{j+k+1}x, \dots, T^{j+p-1}x, T^j x, T^{j+1}x, \dots, T^{j+k-1}x) \\ & = & s_p(T^{pn}x, T^{pn+1}x, \dots, T^{pn-k+p-1}x, T^{pn-k}x, \dots, T^{pn-1}x) \\ & = & s_{p,n,0,k}(x, T^{pn}x) \leq s_{p,n,1,k}(x, T^{pn}x) \\ & = & s_p(T^{pn-1}x, T^{pn}x, \dots, T^{pn-k+p-2}x, T^{pn-k-1}x, \dots, T^{pn-2}x) \\ & = & s_p(T^{j+k-1}x, T^{j+k}x, \dots, T^{j+p-2}x, T^{j-1}x, T^j x, \dots, T^{j+k-2}x) \\ & = & s_p(T^{j-1}x, T^j x, \dots, T^{j+k-2}x, T^{j+k-1}x, T^{j+k}x, \dots, T^{j+p-2}x) \\ & = & r_{j-1}. \end{array}$$

(iii) Put $r_n = s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x)$, then $r_n \geq P$. It follows from (ii) that the sequence $\{r_n\}_{n=1}^{\infty}$ is a nonincreasing sequence. Hence $\lim\limits_{n\to\infty} r_n = r \geq P$.

We claim that $r = P$. Let us suppose the contrary, i.e. $r > P$. Put $\varepsilon_0 = r - P > 0$. There exists $\delta > 0$ such that the inequality $r_n < P + \varepsilon_0$ holds whenever

$$r_{n-1} < P + \varepsilon_0 + \delta. \tag{4}$$

By $\lim\limits_{n\to\infty} r_n = r$ it follow that there is $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ there holds the inequalities $r \leq r_n < r + \delta = \varepsilon_0 + P + \delta$. Therefore (4) holds for $n - 1 \geq n_0$. Thus by the assumption that T is a p–summing cyclic orbital Meir–Keeler contraction of type 2 the inequality $r_n < P + \varepsilon_0 = r$ holds for every $n \geq n_0$, which is a contradiction. Consequently $r = P$.

II) The case $D = 0$ is proven in a similar fashion. □

Corollary 6. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) and T be a p–summing cyclic orbital Meir–Keeler contraction of type 2 with a constant D equal either to P of to zero. Then for any $x \in A_1$ that satisfies (1) and for any $j = 0, 1, 2, \dots, p-1$ there hold $\lim\limits_{n\to\infty} \rho(T^{pn+j}x, T^{pn+j+1}x) = \operatorname{dist}(A_{j+1}, A_{j+2})$ and $\lim\limits_{n\to\infty} \rho(T^{pn+p+j}x, T^{pn+j+1}x) = \operatorname{dist}(A_{j+1}, A_{j+2})$.*

Lemma 6. *Let (X,d) be a complete metric space. Let A_i, $i = 1,\dots,p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map, which is a p–summing cyclic orbital Meir-Keeler contraction of type 2 with a constant $D = 0$. Then for any $x \in A_1$ that satisfies (1) and for any $\varepsilon > 0$*

(i) there exists $N_0 \in \mathbb{N}$ such that for any $m \geq n \geq N_0$ there holds

$$s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < \varepsilon; \tag{5}$$

(ii) there is $N_1 \in \mathbb{N}$ so that the inequalities

$$\rho(T^{pn}x, T^{pm+1}x) < \varepsilon \text{ and } \rho(T^{pm+p-1}x, T^{pn}x) < \varepsilon$$

holds for any $m \geq n \geq N_1$.

Proof. (i) We will prove Lemma 6 by induction on m.

Let $\varepsilon > 0$ be arbitrary. There exists $\delta > 0$, such that condition (1) holds true.

By Lemma 5 there exists $N_1 \in \mathbb{N}$ such that there holds the inequality

$$s_p(T^{pn}x, \dots, T^{pn+j}x, \dots, T^{pn+p-1}x) < \varepsilon$$

for every $n \geq N_1$. From Corollary 6 there exists $N_2 \in \mathbb{N}$, such that for every $n \geq N_2$ there hold $\rho(T^{pn+j-2}x, T^{pn+j-1}x) < \frac{\delta}{2p}$ for $j = 1, 2, \dots, p$. Put $N_0 = \max\{N_1, N_2\}$.

Inequality (5) is true for $m = n \geq N_0$.

Let (5) holds true for some $m \geq n$.

We will prove that (5) holds true for $m + 1$.

Put $S_1 = s_p(T^{pn-1}x, T^{p(m+1)}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p-2}x)$.

By Lemma 5 and the inductive assumption we obtain the inequalities

$$\begin{aligned}
S_1 &= s_p(T^{pn-1}x, T^{p(m+1)}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p+1}x) \\
&\leq s_p(T^{p(n+1)-1}x, T^{p(m+1)}x, \dots, T^{p(m+1)+p-2}) \\
&\quad +2\rho(T^{pn-1}x, T^{p(n+1)-1}x) \\
&\leq s_p(T^{p(n+1)-1}x, T^{p(m+1)}x, \dots, T^{p(m+1)+p-2}) \\
&\quad +2\sum_{j=1}^{p}\rho(T^{pn+j-2}x, T^{pn+j-1}x) \\
&\leq s_p(T^{pn}x, T^{pm+1}x, \dots, T^{pm+p-1}) \\
&\quad +2\sum_{j=1}^{p}\rho(T^{pn+j-2}x, T^{pn+j-1}x) \\
&< \varepsilon + 2p\frac{\delta}{2p} = \varepsilon + \delta.
\end{aligned} \tag{6}$$

The map T is a p–summing cyclic orbital Meir–Keeler contraction of type 2 with $D = 0$ and from the choice of $x \in A_1$, $\delta > 0$ and (6) it follows that

$$s_p(T^{pn}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p-1}x) < \varepsilon.$$

(ii) The proof follows directly from (i). □

Lemma 7. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Let T be a p–summing cyclic orbital Meir–Keeler contraction of type 2 with a constant $D = P$. Then for every $x \in A_1$, satisfying (1) there hold:*

(i) $\lim_{n\to\infty} \|T^{pn+j}x - T^{p(n+1)+j}x\| = 0$ *for every* $j = 0, 1, \dots, p-1$;

(ii) for any $\varepsilon > 0$ *there exists* $N_0 \in \mathbb{N}$ *such that for any* $m \geq n \geq N_0$ *there holds the inequality*

$$s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < P + \varepsilon; \tag{7}$$

(iii) for any $\varepsilon > 0$ *there exists* $N_1 \in \mathbb{N}$ *such that* $\rho(T^{pn}x, T^{pm+1}x) < \varepsilon$ *and* $\rho(T^{pm+p-1}x, T^{pn}x) < \varepsilon$ *hold for any* $m \geq n \geq N_1$.

Proof. (i) By Corollary 6 for any $j = 0, 1, \dots, p-1$ it follows that

$$\lim_{n\to\infty} \|T^{pn+j}x - T^{pn+j+1}x\| = \operatorname{dist}(A_{j+1}, A_{j+2})$$

and

$$\lim_{n\to\infty} \|T^{pn+p+j}x - T^{pn+j+1}x\| = \operatorname{dist}(A_{j+1}, A_{j+2}).$$

According to Lemma 3 it follows $\lim_{n\to\infty} \|T^{pn+j}x - T^{p(n+1)+j}x\| = 0$.

(ii) We will prove by induction on m.

Let $\varepsilon > 0$ be arbitrary. There exists $\delta > 0$, such that condition (1) holds true.

By Lemma 5 there exists $N_1 \in \mathbb{N}$ such that there holds the inequality

$$s_p(T^{pn}x, \dots, T^{pn+j}x, \dots, T^{pn+p-1}x) < P + \varepsilon$$

for every $n \geq N_1$. By (i) there exists $N_2 \in \mathbb{N}$ such that there hold the inequalities $\|T^{pn-p}x - T^{pn}x\| < \delta/2$ for every $n \geq N_2$. Put $N_0 = \max\{N_1, N_2\}$.

Inequality (7) is true for $m = n \geq N_0$.

Let (7) holds true for some $m \geq n$.

We will prove that (7) holds true for $m + 1$.

Let us put $S_2 = s_p(T^{pn-p}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x)$. It is easy to observe that

$$\begin{aligned} S_2 &= \|T^{pn-p}x - T^{pm+1}x\| + \sum_{j=pm+1}^{pm+p-2} \|T^j x - T^{j+1}x\| \\ &\quad + \|T^{pm+p-1}x - T^{pn-p}x\| \\ &\leq \|T^{pn-p}x - T^{pn}x\| + \|T^{pn}x - T^{pm+1}x\| \\ &\quad + \sum_{j=pm+1}^{pm+p-2} \|T^j x - T^{j+1}x\| \\ &\quad + \|T^{pm+p-1}x - T^{pn}x\| + \|T^{pn-p}x - T^{pn}x\| \\ &= s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) \\ &\quad + 2\|T^{pn-p}x - T^{pn}x\|. \end{aligned}$$

Consequently for any $n \geq N_0$ there holds $S_2 \leq P + \varepsilon + \delta$. From (1) we get the inequality

$$\begin{aligned} S_2 &= s_p(T^{pn-p}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) \\ &= s_p(T^{pm+p-1}x, T^{pn-p}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-2}x) \\ &\leq P + \varepsilon + \delta. \end{aligned}$$

Therefore from (1) it follows that

$$s_p(T^{pm+p}x, T^{pn-p+1}x, T^{p(m+1)-p+2}x, \dots, T^{p(m+1)-1}x) < P + \varepsilon.$$

Thus we get

$$s_p(T^{pn-p+1}x, T^{p(m+1)-p+2}x, T^{p(m+1)-p+3}x, \dots, T^{p(m+1)}x) < P + \varepsilon.$$

Put $S_3 = s_p(T^{pn}x, T^{p(m+1)+1}x, T^{p(m+1)+2}x, \dots, T^{p(m+1)+p-1}x)$ and $S_4 = s_p(T^{pn-p+1}x, T^{p(m+1)-p+2}x, T^{p(m+1)-p+3}x, \dots, T^{p(m+1)}x)$.

From Lemma 5 we get the inequalities $S_2 \leq S_4 < P + \varepsilon$.

(iii) The proof follows from (ii). □

4. Proof of Main Result

Proof. (of Theorem 4) (a) Let $x \in A_1$ satisfies (1). We claim that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that the inequality $\rho(T^{pm}x, T^{pn}x) < \varepsilon$ holds for any $m \geq n \geq N_0$.

For any $\varepsilon > 0$ by Lemma 6 there is $N_0 \in \mathbb{N}$ such that there holds the inequality

$$\max\{\rho(T^{pn}x, T^{pm+1}x), \rho(T^{pm+1}x, T^{pm}x)\} < \varepsilon/2$$

for every $m \geq n \geq N_0$. Thus by the inequalities

$$\rho(T^{pn}x, T^{pm}x) \leq \rho(T^{pn}x, T^{pm+1}x) + \rho(T^{pm+1}x, T^{pm}x) < \varepsilon$$

it follows that the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequences and therefore by the completeness of the space (X, ρ) it follows that there exists $\xi \in X$ such that $\lim_{n\to\infty} T^{pn}x = \xi$.

By the inequality $\rho(T^{pn+1}x, \xi) \leq \rho(T^{pn+1}x, T^{pn}x) + \rho(T^{pn}x, \xi)$ and Lemma 6 it follows that

$$\lim_{n\to\infty} T^{pn+1}x = \xi. \tag{8}$$

From the inequality $\rho(T^{pn+2}x, \xi) \leq \rho(T^{pn+2}x, T^{pn+1}x) + \rho(T^{pn+1}x, \xi)$, (9) and Lemma 6 it follows that

$$\lim_{n\to\infty} T^{pn+2}x = \lim_{n\to\infty} T^{pn}x = \lim_{n\to\infty} T^{pn+1}x = \xi. \tag{9}$$

We can obtain in a similar fashion the equalities

$$\lim_{n\to\infty} T^{pn+j}x = \lim_{n\to\infty} T^{pn}x = \xi$$

holds for every $j = 0, 1, 2, \dots, p-1$. Since A_i, $i = 1, 2, \dots p$ are closed sets we obtain that $\xi \in A_i$ for every $i = 1, 2, \dots, p$. Consequently we get that $\xi \in \bigcap_{i=1}^{p} A_i$.

We will prove that $T\xi = \xi$. We apply (5) and the continuity if the function $\rho(\cdot, y)$ in the next chain of inequalities

$$\begin{array}{rcl}
\rho(\xi, T\xi) & \leq & s_p(\xi, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\
& = & \lim_{n\to\infty} s_p(T^{pn}x, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\
& \leq & \lim_{n\to\infty} s_p(T^{pn-1}x, \xi, T\xi, \dots, T^{p-2}\xi) \\
& = & \lim_{n\to\infty} s_p(T^{pn-1}x, T^{pn}x, T\xi, \dots, T^{p-2}\xi) \\
& \leq & \lim_{n\to\infty} s_p(T^{pn-2}x, T^{pn-1}x, \xi, T\xi, \dots, T^{p-3}\xi).
\end{array}$$

By applying the above procedure p–times and Lemma 5 we get

$$\begin{array}{rcl}
\rho(\xi, T\xi) & \leq & s_p(\xi, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\
& \leq & \lim_{n\to\infty} s_p(T^{p(n-1)}x, T^{p(n-1)+1}x, \dots, T^{p(n-1)+(p-1)}x) \\
& = & 0.
\end{array}$$

Thus ξ is a fixed point for the map T.

(b) It remains to prove that ξ is unique.

Suppose that there exists $z \in A_1$, $z \neq x$, which satisfies (1). Then by what we have just proved it follows that $\{T^{pn}z\}_{n=1}^{\infty}$ converges to some point $\eta \in \bigcap_{i=1}^{p} A_i$, such that $T\eta = \eta$. Since $D = 0$ it follows that

$$\lim_{n\to\infty} s_p(T^{pn}z, T^{pn+1}x, T^{pn+2}x, \dots, T^{pn+p-1}x) = 0.$$

From the continuity of the function $\rho(\cdot,\cdot)$ and Lemma 5 we get

$$\begin{array}{rcl} \rho(\eta,\xi) & = & \lim\limits_{n\to\infty} \rho(T^{pn}z, T^{pn+1}x) \\ & \leq & \lim\limits_{n\to\infty} s_p(T^{pn}z, T^{pn+1}x, T^{pn+2}x, \dots, T^{pn+p-1}x) = 0. \end{array}$$

Hence $\xi = \eta$. □

Proof. (of Theorem 5) Let $x \in A_1$ satisfies (1).

Case I) Let $D = 0$. From Theorem 4 there exists a unique fixed point of T, which is a best proximity point.

Case II) (a) Let $D = P > 0$. We will prove that the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequence. By Corollary 6 we have that $\lim\limits_{m\to\infty} \|T^{pm}x - T^{pm+1}x\| = \operatorname{dist}(A_1, A_2)$. From Lemma 7 we have that for any $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$, such that there holds the inequality

$$s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < P + \varepsilon/2$$

for every $m \geq n \geq N_1$. Thus $\|T^{pn}x - T^{pm+1}x\| \leq \operatorname{dist}(A_1, A_2) + \varepsilon/2$ holds for every $m \geq n \geq N_1$. According to Lemma 2 it follows that for any $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$, such that for any $m \geq n \geq N_2$ there holds the inequality $\|T^{pn}x - T^{pm}x\| \leq \varepsilon/2 < \varepsilon$ and thus $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent to some $\xi \in A_1$.

(b) By Lemma 5 and the continuity of the function $\|\cdot\|$ we can write the chain of inequalities

$$\begin{array}{rcl} P & \leq & s_p(\xi, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\ & = & \lim\limits_{n\to\infty} s_p(T^{pn}x, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\ & \leq & \lim\limits_{n\to\infty} s_p(T^{pn-1}x, \xi, T\xi, \dots, T^{p-2}\xi) \\ & = & \lim\limits_{n\to\infty} s_p(T^{pn-1}x, T^{pn}x, T\xi, \dots, T^{p-2}\xi) \\ & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \leq & \lim\limits_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}x, T^{pn-p+2}x, \dots, T^{pn-1}x) \\ & = & P. \end{array} \tag{10}$$

Form (10) we get that $\|\xi - T\xi\| = \mathrm{dist}(A_1, A_2)$, $\|\xi - T^{p-1}\xi\| = \mathrm{dist}(A_1, A_p)$, $\|T^j\xi - T^{j+1}\xi\| = \mathrm{dist}(A_{j+1}, A_{j+2})$, $j = 1, 2, \dots p-2$. Thus ξ is a best proximity point of T in A_1, $T^j\xi$, $j = 1, 2, \dots p-1$ is a best proximity point of T in A_{j+1}.

It remains to show that the point ξ from (a) and (b) is unique. We will show that for any $z \in A_1$, $z \neq x$, such that z satisfies (1) there holds $\lim_{n\to\infty} T^{pn}z = \xi$. By what we have just proved $\{T^{pn}z\}$ converges to a best proximity point, say $\eta \in A_1$, of T in A_1. From Lemma 5

$$\lim_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}z, T^{pn-p+2}z, \dots, T^{pn-1}z) = P. \tag{11}$$

By Lemma 5, the continuity of the function $\|\cdot\|$ and (11) we get

$$\begin{array}{rcl}
P & \leq & s_p(\xi, T\eta, T^2\eta, \dots, T^{p-1}\eta) \\
 & = & \lim_{n\to\infty} s_p(T^{pn}x, T\eta, T^2\eta, \dots, T^{p-1}\eta) \\
 & \leq & \lim_{n\to\infty} s_p(T^{pn-1}x, \eta, T\eta, \dots, T^{p-2}\eta) \\
 & = & \lim_{n\to\infty} s_p(T^{pn-1}x, T^{pn}z, T\eta, \dots, T^{p-2}\eta) \\
 & \leq & \lim_{n\to\infty} s_p(T^{pn-2}x, T^{pn-1}z, \eta, T\eta, \dots, T^{p-3}\eta) \\
 & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
 & \leq & \lim_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}z, T^{pn-p+2}z, \dots, T^{pn-1}z) = P.
\end{array}$$

Therefore we get that $\|\xi - T\eta\| = \|\xi - T\xi\| = \mathrm{dist}(A_1, A_2)$. Since A_2 is convex set in a uniformly convex Banach space it follows that $T\eta = T\xi$. By the fact that η is a best proximity point of T in A_1 there hold the equalities $\|\eta - T\eta\| = \|\eta - T\xi\| = \mathrm{dist}(A_1, A_2) = \|\xi - T\xi\|$. Since A_1 a convex set in a uniformly convex Banach space and $T\eta = T\xi$ it follows that $\eta = \xi$.

(c) From the inequality $\|T^{pn+1}x - \xi\| \leq \|T^{pn+1}x - T^{pn}x\| + \|T^{pn}x - \xi\|$ it follows the equality $\lim_{n\to\infty} \|T^{pn+1}x - \xi\| = \mathrm{dist}(A_1, A_2)$. By Lemma 5 and $\|T\xi - \xi\| = \mathrm{dist}(A_1, A_2)$ we get $\lim_{n\to\infty} T^{pn+1}x = T\xi$. From the inequality $\|T^{pn+2}x - T\xi\| \leq \|T^{pn+2}x - T^{pn+1}x\| + \|T^{pn+1}x - T\xi\|$ it follows that $\lim_{n\to\infty} \|T^{pn+1}x - T\xi\| = \mathrm{dist}(A_2, A_3)$. By Lemma 5 and $\|T^2\xi - T\xi\| = \mathrm{dist}(A_2, A_3)$ we get $\lim_{n\to\infty} T^{pn+2}x = T^2\xi$. By continuing this we can prove $\lim_{n\to\infty} T^{pn+k}x = T^k\xi$.

Thus we can write the chain of inequalities

$$\begin{array}{rcl} P & \le & s_p(\xi, T^{p+1}\xi, T^{p+2}\xi, \dots, T^{2p-1}\xi) \\ & = & \lim\limits_{n\to\infty} s_p(T^{pn}x, T^{p+1}\xi, T^{p+2}\xi, \dots, T^{2p-1}\xi) \\ & \le & \lim\limits_{n\to\infty} s_p(T^{pn-1}x, T^{p}\xi, T^{p+1}\xi, \dots, T^{2p-2}\xi) \\ & = & \lim\limits_{n\to\infty} s_p(T^{pn-1}x, T^{pn}x, T^{p+1}\xi, \dots, T^{2p-2}\xi) \\ & \le & \lim\limits_{n\to\infty} s_p(T^{pn-2}x, T^{pn-1}x, T^{p}\xi, \dots, T^{2p-3}\xi) \\ & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \le & \lim\limits_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}x, T^{pn-p+2}x, \dots, T^{pn-1}x) = P. \end{array}$$

By the above inequality we get $d(\xi, T^{p+1}\xi) = \operatorname{dist}(A_1, A_2)$ and from (b) we have $d(T^p\xi, T^{p+1}\xi) = \operatorname{dist}(A_1, A_2)$. Therefore from Lemma 4 we get $T^p\xi = \xi$. □

5. Applications and Examples

All the examples in [18] can solved with Theorem 5. We illustrate the applications of Theorem 5 with an example, which involves integral operators.

Let us consider the Hilbert space $L_2[-1,1]$, endowed with the norm $\|f\|_2 = \left(\int_{-1}^1 f^2(t)dt\right)^{\frac{1}{2}}$ and the functions $x_i \in L_2[-1,1]$, $i = 1,2,3,4$, that satisfy $x_1(t) = 0$ for $t \in [-1,0]$ and $x_1(t) > 0$ for $t \in (0,1]$; $x_2(t) = 0$ for $t \in [0,1]$ and $x_2(t) > 0$ for $t \in [-1,0)$; $x_3(t) = 0$ for $t \in [-1,0]$ and $x_3(t) < 0$ for $t \in (0,1]$; $x_4(t) = 0$ for $t \in [0,1]$ and $x_4(t) < 0$ for $t \in [-1,0)$. It is well known that any Hilbert space is uniformly convex. We will consider the sets A_i, $i = 1,2,3,4$, defined by

$$A_i = \{f \in L_2[-1,1] : f(t) \ge x_i(t) \text{ for all } t \in [-1,1]\} \text{ for } i = 1,2$$

and

$$A_i = \{f \in L_2[-1,1] : f(t) \le x_i(t) \text{ for all } t \in [-1,1]\} \text{ for } i = 3,4.$$

It is easy to calculate that

$$\begin{array}{rcl} P & = & \operatorname{dist}(A_1, A_2) + \operatorname{dist}(A_2, A_3) + \operatorname{dist}(A_3, A_4) + \operatorname{dist}(A_4, A_1) \\ & = & \|x_1\|_2 + \|x_2\|_2 + \|x_3\|_2 + \|x_4\|_2. \end{array}$$

Let us denote the maps $T_i : L_{[-1,1]} \to L_{[-1,1]}$, $i = 1,2,3,4$ as follows:

$$(T_i y)(t) = \begin{cases} f_i(t) + \int_0^1 F_i(t, s, y(s))ds, & \text{for } t \in [0,1] \\ 0, & \text{for } t \in [-1,0], \end{cases}$$

where $f_i \in L_{[0,1]}$, $i = 1,2,3,4$ and $F_i : [0,1] \times [0,1] \times L_{[0,1]} \to \mathbb{R}$, $i = 1,2,3,4$ be continuous functions, such that for $v_i \in A_i$ there hold the inequalities $(T_1 v_1(s))(-t) \geq x_2(t)$, $(-T_2 v_2(-s))(t) \leq x_3(t)$, $(T_3(-v_3(s)))(-t) \leq x_4(t)$ and $(T_4(-v_4(-s)))(t) \geq x_1(t)$ for any $t \in [-1,1]$. Let us define a 4–cyclic map $T : A_i \to A_{i+1}$ by

$$(Tx(s))(t) = \begin{cases} (T_1 x(s))(-t), & x \in A_1 \\ -(T_2 x(-s))(t), & x \in A_2 \\ (T_3(-x(s)))(-t), & x \in A_3 \\ -T_4(-x(-s))(t), & x \in A_4. \end{cases} \tag{12}$$

The next theorem is a direct consequence of Theorem 5

Theorem 7. *Let $T : \cup_{i=1}^4 A_i \to \cup_{i=1}^p A_i$ be the map defined in (12). If T is a 4–summing cyclic orbital Meir–Keeler contraction of type 2 with constant $D = P$, then there exists a unique point, say $\xi \in A_1$, such that:*

(a) for every $x \in A_1$ satisfying (1)*, the sequence $\{T^{4n}x\}$ converges to ξ;*

(b) ξ is a best proximity point of T in A_1 and $T^j\xi$ is a best proximity point of T in A_{1+j};

(c) ξ is a fixed point for the map $T^4 : A_1 \to A_1$.

We will present a particular example of Theorem 7.

Example: Let the functions $x_i \in L_2[-1,1]$, $i = 1,2,3,4$ be defined by $x_1 = \begin{cases} 0, & -1 \leq t \leq 0 \\ t, & 0 \leq t \leq 1 \end{cases}$; $x_2(t) = 2x_1(-t)$; $x_3(t) = -2x_1(t)$ and $x_4(t) = -x_1(-t)$. It is easy to calculate that $P = \sum_{i=1}^4 \operatorname{dist}(A_i, A_{i+1}) = \left(\frac{2}{3}\right)^{1/2} + 2\left(\frac{5}{3}\right)^{1/2} + \left(\frac{8}{3}\right)^{1/2}$.

Let us define $F_1(t,s,x(s)) = tsx(s)$, $F_i(t,s,x(s)) = \frac{ts}{2}x(s)$ for $i = 2,3,4$, $f_i(t) = \frac{5t}{3}$ for $i = 1,2$, $f_3(t) = \frac{2t}{3}$ and $f_4(t) = \frac{5t}{6}$. Then the functions f_i, F_i for $i = 1,2,3,4$ satisfy the conditions of Theorem 7 and $T : A_i \to A_{i+1}$ is a 4–cyclic map.

We will prove T satisfies Definition 3 with $p = 4$ and $x = x_1(s)$ for $k = 1$, i.e.: if $s_{4,n,1,1}(x,y) < P + \varepsilon + \delta$ for $n \in \mathbb{N}$ and $y \in A_1$ then there holds the inequality $s_{4,n,0,1}(x,y) < P + \varepsilon$.

The prove for $k = 2,3$ can be done in a similar fashion.

It is easy to observe that $(T^{4n}x)(t) = x(t)$, $(T^{4n-1}x)(t) = x_4(t)$, $(T^{4n-2}x)(t) = x_3(t)$ and $(T^{4n-3}x)(t) = x_2(t)$ for every $n \in \mathbb{N}$.

Let $y \in A_1$. There exists a function $\alpha : [-1,1] \to [0,+\infty)$, which satisfies $\alpha(t) \geq 0$ for $t \in [0,1]$ and $\alpha(t) = 0$ for $t \in [-1,0]$, such that $y(t) = t + \alpha(t)$. Let us denote $c = c(\alpha) = \int_0^1 s\alpha(s)ds$. It is easy to see that

$$\begin{array}{rcl} Ty(s)(t) & = & (T_1 y(s))(t) \\ & = & 2\left(\frac{5t}{6} + \int_0^1 \frac{ts}{2}(s+\alpha(s))ds\right) = t(2+c); \\ T^2y(s)(t) & = & (T_2T_1y(s))(t) = (T_2(s(2+c)))(t) \\ & = & \frac{5t}{3} + \int_0^1 \frac{ts}{2}(s(2+c))ds = t\left(2+\frac{c}{6}\right) \\ T^3y(s)(t) & = & (T_3T_2T_1y(s))(t) = \left(T_3\left(s\left(2+\frac{c}{6}\right)\right)\right)(t) \\ & = & \frac{2t}{3} + \int_0^1 \frac{ts}{2}s\left(2+\frac{c}{6}\right)ds = t\left(1+\frac{c}{36}\right). \end{array}$$

and after some calculations we get

$$\begin{array}{rcl} \|T^{4n-1}x - y\|_2 & = & \left(\int_{-1}^0 t^2dt + \int_0^1 (t+\alpha(t))^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{2}{3} + 2c + \int_0^1 \alpha^2(t)dt\right)^{\frac{1}{2}}; \\ \|y - Ty\|_2 & = & \left(\int_0^1 (t+\alpha(t))^2dt + \int_{-1}^0 t^2(2+c)^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{5}{3} + \frac{10c}{3} + \frac{c^2}{3} + \int_0^1 \alpha^2(t)dt\right)^{\frac{1}{2}}; \\ \|Ty - T^2y\|_2 & = & \left(\int_{-1}^0 t^2(2+c)^2dt + \int_0^1 t^2\left(2+\frac{c}{6}\right)^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{8}{3} + \frac{14c}{9} + \frac{37c^2}{108}\right)^{1/2}; \\ \|T^2y - T^{4n-1}x\|_2 & = & \left(\int_0^1 t^2\left(2+\frac{c}{6}\right)^2dt + \int_{-1}^0 t^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{5}{3} + \frac{2c}{9} + \frac{c^2}{108}\right)^{\frac{1}{2}}; \\ \|T^{4n}x - Ty\|_2 & = & \left(\int_0^1 t^2dt + \int_{-1}^0 t^2(2+c)^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{5}{3} + \frac{4c}{3} + \frac{c^2}{3}\right)^{\frac{1}{2}}; \\ \|T^2y - T^3y\|_2 & = & \left(\int_0^1 t^2\left(2+\frac{c}{6}\right)^2dt + \int_{-1}^0 t^2\left(1+\frac{c}{36}\right)^2dt\right)^{\frac{1}{2}} \\ & = & \left(\frac{5}{3} + \frac{13c}{54} + \frac{37c^2}{3888}\right)^{\frac{1}{2}}; \end{array}$$

$$
\begin{aligned}
\|T^3y - T^{4n}x\|_2 &= \left(\int_{-1}^{0} t^2\left(1+\frac{c}{36}\right)^2 dt + \int_0^1 t^2 dt +\right)^{\frac{1}{2}} \\
&= \left(\frac{2}{3}+\frac{c}{54}+\frac{c^2}{3888}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
f(\alpha(t)) &= s_4(T^{4n-1}x, y, Ty, T^2y) = \left(\tfrac{2}{3} + 2c + \textstyle\int_0^1 \alpha^2(t)dt\right)^{\frac{1}{2}} \\
&\quad + \left(\tfrac{5}{3} + \tfrac{10c}{3} + \tfrac{c^2}{3} + \textstyle\int_0^1 \alpha^2(t)dt\right)^{\frac{1}{2}} + \left(\tfrac{8}{3} + \tfrac{14c}{9} + \tfrac{37c^2}{108}\right)^{\frac{1}{2}} \\
&\quad + \left(\tfrac{5}{3} + \tfrac{2c}{9} + \tfrac{c^2}{108}\right)^{\frac{1}{2}};
\end{aligned}
$$

$$
\begin{aligned}
h(c) &= s_4(T^{4n}x, Ty, T^2y, T^3y) = \left(\tfrac{2}{3} + \tfrac{c}{54} + \tfrac{c^2}{3888}\right)^{\frac{1}{2}} \\
&\quad + \left(\tfrac{5}{3} + \tfrac{4c}{3} + \tfrac{c^2}{3}\right)^{\frac{1}{2}} + \left(\tfrac{8}{3} + \tfrac{14c}{9} + \tfrac{37c^2}{108}\right)^{\frac{1}{2}} \\
&\quad + \left(\tfrac{5}{3} + \tfrac{13c}{54} + \tfrac{37c^2}{3888}\right)^{\frac{1}{2}};
\end{aligned}
$$

$$
\begin{aligned}
g(c) &= \left(\tfrac{2}{3} + 2c + 3c^2\right)^{\frac{1}{2}} + \left(\tfrac{5}{3} + \tfrac{10c}{3} + \tfrac{10c^2}{3}\right)^{\frac{1}{2}} + \left(\tfrac{8}{3} + \tfrac{14c}{9} + \tfrac{37c^2}{108}\right)^{\frac{1}{2}} \\
&\quad + \left(\tfrac{5}{3} + \tfrac{2c}{9} + \tfrac{c^2}{108}\right)^{\frac{1}{2}}.
\end{aligned}
$$

From the inequality $c^2 \le \left(\int_0^1 t^2 dt\right)\left(\int_0^1 \alpha^2(t)dt\right) = \frac{1}{3}\left(\int_0^1 \alpha^2(t)dt\right)$ we get that there holds the inequality $g(c) \le f(\alpha(t))$. It is easy to see that the functions g and h are strictly increasing functions for $c \in [0, +\infty)$. By the inequalities

$$
\begin{aligned}
I_1(c) &= \left(\tfrac{5}{3} + \tfrac{10c}{3} + \tfrac{10c^2}{3}\right)^{\frac{1}{2}} + \left(\tfrac{5}{3} + \tfrac{2c}{9} + \tfrac{c^2}{108}\right)^{\frac{1}{2}} \\
&= \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left[\left(1 + 2c + 2c^2\right)^{\frac{1}{2}} + \left(1 + \tfrac{2c}{15} + \tfrac{c^2}{180}\right)^{\frac{1}{2}}\right] \\
&\ge \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left[\left(1 + 2c + c^2\right)^{\frac{1}{2}} + \left(1 + \tfrac{2c}{15} + \tfrac{c^2}{15^2}\right)^{\frac{1}{2}}\right] \\
&= \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left(1 + c + 1 + \tfrac{c}{15}\right) = \left(\tfrac{5}{3}\right)^{1/2}\left(2 + \tfrac{16c}{15}\right)
\end{aligned}
$$

and

$$\begin{aligned}
I_2(c) &= \left(\tfrac{5}{3}+\tfrac{4c}{3}+\tfrac{c^2}{3}\right)^{\frac{1}{2}}+\left(\tfrac{5}{3}+\tfrac{13c}{54}+\tfrac{37c^2}{3888}\right)^{\frac{1}{2}}\\
&= \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left[\left(1+\tfrac{4c}{5}+\tfrac{c^2}{5}\right)^{\frac{1}{2}}+\left(1+\tfrac{13c}{90}+\tfrac{37c^2}{6480}\right)^{\frac{1}{2}}\right]\\
&\le \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left[\left(1+\tfrac{2c}{\sqrt{5}}+\tfrac{c^2}{5}\right)^{\frac{1}{2}}+\left(1+2\tfrac{\sqrt{37}}{\sqrt{6480}}c+\tfrac{37c^2}{6480}\right)^{\frac{1}{2}}\right]\\
&= \left(\tfrac{5}{3}\right)^{\frac{1}{2}}\left(1+\tfrac{c}{\sqrt{5}}+1+\tfrac{\sqrt{37}}{\sqrt{6480}}c\right)\\
&= \left(\tfrac{5}{3}\right)^{1/2}\left(2+\left(1+\tfrac{\sqrt{37}}{36}\right)\tfrac{c}{\sqrt{5}}\right)
\end{aligned}$$

it follows that for every $c > 0$ there holds the inequality

$$I_1(c) > \left(\tfrac{5}{3}\right)^{1/2}\left(2+\tfrac{16c}{15}\right) > \left(\tfrac{5}{3}\right)^{1/2}\left(2+\left(1+\tfrac{\sqrt{37}}{36}\right)\tfrac{c}{\sqrt{5}}\right) > I_2(c)$$

and consequently $h(c) < g(c)$ holds for every $c > 0$.

Let $\varepsilon > 0$ be arbitrary chosen. From the equality $h(0) = P$ and the fact that h is strictly increasing in the interval $[0, +\infty)$ it follows that there exists a unique c_0, such that $h(c_0) = P + \varepsilon$ and $h(c) < P + \varepsilon$ for every $c \in [0, c_0)$. Let us put $\delta(\varepsilon) = g(c_0) - h(c_0) > 0$. Now if α be such that $f(\alpha(t)) < P + \varepsilon + \delta(\varepsilon)$ then from the inequality $g(c) \le f(\alpha) < P + \varepsilon + \delta(\varepsilon) = P + \varepsilon + g(c_0) - h(c_0) = g(c_0)$ and the fact that g is strictly increasing it follows that $c < c_0$. Therefore $h(c) < h(c_0) = P + \varepsilon$, because h is an increasing function. Consequently T is a 4–summing cyclic orbital Meir-Keeler contraction of type 2. From Theorem 5 it follows that there exists a unique point $\xi \in A_1$, such that: ξ is a best proximity point of T in A_1 and $T^j\xi$ is a best proximity point of T in A_{1+j} for any j=1,2,3.

All the results in complete metric spaces [7], [8], [18], [10], where the constant D in (1) is zero are covered by Theorem 4.

For other examples involving Integral operators we refer to [10] and [19].

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References

[1] Banach S, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.* **3** (1922) 133–181

[2] DiBari C, Suzuki T and Vetro C, Best proximity points for cyclic Meir–Keeler contractions, *Nonlinear Anal.* **69**(11) (2008) 3790–3794

[3] Eldred A and Veeramani P, Existence and convergence of best proximity points, *J. Math. Anal. Appl.* **323**(2) (2006) 1001–1006

[4] Karapinar E and Erhan I M, Best proximity point on different type contractions, *Applied Mathematics and Information Sciences* **5**(3) (2011) 517–528

[5] Karapinar E, Petrusal G and Tas K, Best proximity point theorems for KT–types Cyclic orbital contraction mappings, *Fixed Point Theory* **13**(2) (2012) 537–546

[6] Karapinar E, Romaguera S and Tas K, Fixed points for cyclic orbital generalized contractions on complete metric spaces, *Cent. Eur. J. Math.* **11**(3) (2013) 552–560

[7] Karpagam S and Sushama Agrawal, Best proximity point for p–cyclic Meir–Keeler contractions, *Fixed Point Theory Appl.* 29, Article ID 197308, 9 pages (2009).

[8] Karpagam S and Sushama Agrawal, Best proximity point theorems for cyclic orbital Meir–Keeler contraction maps, *Nonlinear Anal.* **74**(4) (2011) 1040–1046

[9] Karpagam S and Sushama Agrawal, Existence of best proximity points for p–cyclic contractions, *Fixed Point Theory* **13**(1) (2012) 99–105.

[10] Karpagam S and Zlatanov B, Best proximity points for p–cyclic orbital Meir–Keeler contractions *Nonlinear Anal. Model. Control.* (to appear)

[11] Lim T, On charecterizations of Meir–Keeler contractive maps, *Nonlinear Anal.* **46**(1) (2001) 113–120.

[12] Meir A and Keeler E, A theorem on contractive mappings, *J. Math. Anal. Appl.* **28** (1969) 326–329

[13] Ming Chen C, Fixed point theorems for cyclic Meir–Keeler type mappings in complete metric spaces, *Fixed Point Theory Appl.* 2012:41 (2012)

[14] Petric M and Zlatanov B, Best proximity points and fixed points for p–summing maps, *Fixed Point Theory Appl* 2012:86 (2012)

[15] Petric M,

[16] Ph.D. Thesis: Fixed Points and Best Proximity Points Theorems for Cyclical Contractive Operators, North University of Baia Mare, Faculty of Science (2011)

[17] Suzuki T, Some notes on Meir-Keeler contractions and L–functions, *Bull. Kyushu Inst. Technol. Pure Appl. Math.* **53** (2006) 1–13

[18] Zlatanov B, Best proximity points for p–summing cyclic orbital Meir–Keeler contractions, *Nonlinear Anal. Model. Control.* **20**(4) (2015) 528-544.

[19] Zlatanov B, Best proximity points in modular function spaces, *Arab. J. Math.* **4**(3) (2015) 215-227.