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# $L_p$-EQUIVALENCE BETWEEN TWO NONLINEAR IMPULSE DIFFERENTIAL EQUATIONS WITH UNBOUNDED LINEAR PARTS AND ITS APPLICATION FOR PARTIAL IMPULSE DIFFERENTIAL EQUATIONS

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**Abstract.** An $L_p$-equivalence between two impulse differential equations with unbounded linear parts is proved by means of the Schauder-Tychonoff's fixed point theorem. An example of the theory of the partial impulse differential equations of parabolic type is given.

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**Key words:** Impulse differential equations, Lp-equivalence, Partial impulse differential equations of parabolic type

## 1. Introduction

We study an equivalence in $L_p$ $(1 \leq p < \infty)$ between two ordinary impulse differential equations with a possibly unbounded linear part. This means that to every bounded solution of the first equation there corresponds a bounded solution to the second equation such that their difference is in $L_p$ and vice versa. In Theorem 1 we prove the $L_p$-equivalence making use of the Schauder-Tychonoff's fixed point principle. Further we give an example with an important application in physics. We consider two partial impulse differential equations with elliptic linear parts and reduce them to two ordinary impulse differential equations. These equations satisfy the conditions of Theorem 1 and are therefore $L_p$-equivalent. In this case, we establish "$L_p$-dependence" between the solutions of two partial equations.

## 2. Statement of the problem

Let $X$ be a Banach space with norm $\|.\|$ and identity $I$.

By $D(T) \subset X$ we will denote the domain of the operator $T : D(T) \to X$.

We consider the following two impulse differential equations

$$\frac{du_i}{dt} = A_i(t)u_i + f_i(t, u_i) \quad \text{for} \;\; t \neq t_n \tag{1}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) + h_n^i(u_i(t_n)) \quad \text{for} \;\; n = 1, 2, ..., \tag{2}$$

where $A_i(t) : D(A_i(t)) \to X$ $(t \in \mathbb{R}_+)$ and $Q_n^i : D(Q_n^i) \to D(A_i(t_n))$ $(i = 1, 2)$ are linear (possibly unbounded) operators. The sets $D(A_i(t))$ and $D(Q_n^i)$ $(i = 1, 2;\ t \geq 0,\ n = 1, 2, ...)$ are dense in $X$. The functions $f_i(.,.) : \mathbb{R}_+ \times X \to X$ and $h_n^i : X \to X (n = 1, 2, ...)$ are continuous. The points of jump $t_n$ satisfy the following conditions $0 = t_o < t_1 < ... < t_n < ...,\ \lim\limits_{n\to\infty} t_n = \infty$. We set $Q_0^i = I,\ h_0^i(u) = 0\ \ (i = 1, 2,\ u \in X)$.

Furthermore, we assume that all considered functions are left continuous.

Let $U_i(t, s)$ $(i = 1, 2;\ 0 \leq s \leq t)$ be Cauchy operators of the linear ordinary equations

$$\frac{du_i}{dt} = A_i(t)u_i \quad (i = 1, 2). \tag{3}$$

It is easy to prove that the functions $u_i(t) = V_i(t, s)\xi_i$ for $\xi_i \in D(A_i(s))$ $(i = 1, 2)$ with

$$V_i(t, s) = U_i(t, t_n)Q_n^i U_i(t_n, t_{n-1})Q_{n-1}^i ... Q_k^i U_i(t_k, s) \tag{4}$$

$(0 \leq s \leq t_k \leq t_n < t)$ satisfy the linear impulse Cauchy problems

$$\frac{du_i}{dt} = A_i(t)u_i \quad \text{for} \;\; t \neq t_n \tag{5}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) \quad \text{for} \;\; n = 1, 2, ... \tag{6}$$

$$u_i(s) = \xi_i \quad (i = 1, 2). \tag{7}$$

Let us note that the operators $V_i(t, s)$ $(i = 1, 2)$ are bounded if one of the following conditions holds.

1. $Q_n^i U_i(t_n, t_{n-1})$ are bounded operators $(i = 1, 2;\ n = 1, 2, ...)$.
2. $U_i(t_{n+1}, t_n)Q_n^i$ are bounded operators $(i = 1, 2;\ n = 1, 2, ...)$.

**Definition 1.** *The solutions of integral equations*

$$u_i(t) = V_i(t,s)\xi_i + \int\limits_s^t V_i(t,\tau)f_i(\tau,u_i(\tau))d\tau + \sum_{s<t_n<t} V_i(t,t_n^+)h_n^i(u_i(t_n)) \tag{8}$$

*for* $0 \le s \le t,\ \xi_i \in D(A_i(s)),\ u_i(s) = \xi_i$ *are called solutions of the impulse equations* $(1),(2)$ $(i = 1,2)$.

By $L_p(X),\ 1 \le p < \infty$ we denote the space of all functions $u : \mathbb{R}_+ \to X$ for which $\int\limits_0^\infty \|u(t)\|^p dt < \infty$ with norm $\|u\|_p = (\int\limits_0^\infty \|u(t)\|^p dt)^{\frac{1}{p}}$. Set $B_r = \{u \in X :\ \|u\| \le r\}$.

**Definition 2.** *The equation* $(1),(2)$ *for* $i = 2$ *is called* $L_p$*-equivalent to the equation* $(1),(2)$ *for* $i = 1$ *in the ball* $B_r$*, if there exists* $\rho > 0$ *such that for any solution* $u_1(t)$ *of* $(1),(2)$ $(i = 1)$ *lying in the ball* $B_r$ *there exists a solution* $u_2(t)$ *of* $(1),(2)$ $(i = 2)$ *lying in the ball* $B_{r+\rho}$ *and satisfying the relation* $u_2(t) - u_1(t) \in L_p(X)$. *If equation* $(1),(2)$ $(i = 2)$ *is* $L_p$*-equivalent to equation* $(1),(2)$ $(i = 1)$ *in the ball* $B_r$ *and vice versa, we shall say that equations* $(1),(2)$ $(i = 1)$ *and* $(1),(2)$ $(i = 2)$ *are* $L_p$*-equivalent in the ball* $B_r$.

The paper aims at finding sufficiently conditions for the existence of $L_p$-equivalence between the impulse equations $(1),(2)$ $(i = 1,2)$.

## 3. Main results

### 3.1. $L_p$-equivalent impulse equations

Let us set

$$v(t) = u_2(t) - u_1(t),$$

$u_i(t)$ $(i = 1,2)$ being defined by (8).

Then the function $v(t)$ is a solution of the integral equation

$$v(t) = T(u_1,v)(t),$$

where

$$\begin{aligned} T(u_1,v)(t) &= V_2(t,0)(u_1(0)+v(0)) - V_1(t,0)u_1(0)+ \\ &+ \int\limits_0^t \{V_2(t,\tau)f_2(\tau,u_1(\tau)+v(\tau)) - V_1(t,\tau)f_1(\tau,u_1(\tau))\}d\tau+ \\ &+ \sum_{0<t_n<t} \{V_2(t,t_n^+)h_n^2(u_1(t_n)+v(t_n)) - V_1(t,t_n^+)h_n^1(u_1(t_n))\} \end{aligned} \tag{9}$$

We shall prove that for each solution $u_1(t)$ of equation (1), (2) $(i = 1)$ lying in the ball $B_r$ the operator $T(u_1, v)$ has a fixed point $v(t)$ such that $u_1(t) + v(t) \in B_{r+\rho}$ for some $\rho > 0$ and which is in $L_p(X)$.

Let $S(\mathbb{R}_+, X)$ be linear set of all functions which are continuous for $t \neq t_n$ ($n$=1, 2, ...), have both left and right limits at points $t_n$ and are left continuous. The set $S(\mathbb{R}_+, X)$ is a locally convex space w.r.t. the metric

$$\rho(u, v) = \sup_{0<T<\infty} (1+T)^{-1} \frac{\max\limits_{0\le t\le T} \|u(t) - v(t)\|}{1 + \max\limits_{0\le t\le T} \|u(t) - v(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

**Lemma 1.** [1] *The set $M \subset S(\mathbb{R}_+, X)$is relatively compact if and only if the intersections $M(t) = \{m(t) : \ m \in M\}$ are relatively compact for $t \in \mathbb{R}_+$ and $M$ is equicontinuous on each interval $(t_n, t_{n+1}]$ $(n = 0, 1, 2, ...)$.*

**Proof.** We apply Arzella-Ascoli theorem to each interval $(t_n, t_{n+1}]$ $(n = 0, 1, 2, ...)$ and constitute a diagonal line sequence, which is converging on each of them. □

**Lemma 2.** [1] *Let the continuous compact operator $T$ transform the set*

$$C(\rho) = \{v \in S(\mathbb{R}_+, X) : \ v(t) \in B_\rho, \ t \in \mathbb{R}_+\}$$

*onto itself.*

*Then $T$ has a fixed point in $C(\rho)$.*

### 3.2. Conditions for $L_p$-equivalence

**Theorem 1.** *Let the following conditions be fulfilled.*

1. *There exist positive functions $K_i(t, s)$ $(i = 1, 2)$ such that*

$$\|V_i(t, s)\xi\| \le K_i(t, s)\|\xi\| \quad (0 \le s \le t, \ \xi \in D(A_i(s))),$$

*where the functions $K_i(t, 0)$ $(i = 1, 2)$ satisfy the following condition.*

*There exist constants $r, \rho > 0$ such that*

$$K_1(t, 0)\|\xi\| + K_2(t, 0)\|\eta\| \le \chi_{r,\rho}(t) \quad (t \in \mathbb{R}_+, \ \eta \in B_{r+\rho}, \ \xi \in B_r),$$

*where $\chi_{r,\rho}(t) \in L_p(\mathbb{R}_+)$.*

2. *The functions* $f_i(t,u)$ *and* $K_i(t,s)$ $(i=1,2)$ *satisfy the conditions.*

2.1 $\sup\limits_{\|u\|\le r}\int\limits_0^t K_1(t,\tau)\|f_1(\tau,u)\|d\tau+\sup\limits_{\|w\|\le r+\rho}\int\limits_0^t K_2(t,\tau)\|f_2(\tau,w)\|d\tau\le\psi_{r,\rho}(t)$,
*where* $\psi_{r,\rho}(t)$ *is continuous and* $\psi_{r,\rho}(t)\in L_p(\mathbb{R}_+)$.

2.2 $\int\limits_0^t V_2(t,\tau)f_2(\tau,u_1(\tau)+v(\tau))d\tau\in K(t)$
$(v\in B_\rho,\ u_1\in B_r,\ u_1-$ *fixed*$)$, *where for any fixed* $t\in\mathbb{R}_+$ $K(t)$ *is a compact subset of* $X$.

3. *The functions* $h_n^i(u)$ *and* $K_i(t,s)$ $(i=1,2)$ *satisfy the conditions.*

3.1 $\sup\limits_{\|u\|\le r}\sum\limits_{0<t_n<t}K_1(t,t_n^+)\|h_n^1(u)\| + \sup\limits_{\|w\|\le r+\rho}\sum\limits_{0<t_n<t}K_2(t,t_n^+)\|h_n^2(w)\|\le$
$\le\varphi_{r,\rho}(t)$, *where* $\varphi_{r,\rho}(t)\in L_p(\mathbb{R}_+)$.

3.2 $\sum\limits_{0<t_n<t}V_2(t,t_n^+)h_n^2(u_1(t_n)+v(t_n))\in K_n$
$(v\in B_\rho,\ u_1\in B_r,\ u_1-$ *fixed*$)$, *where for any fixed* $n=1,2,...$ $K_n$ *is a compact subsets of* $X$.

4. *The function* $f_2(t,w)$ *satisfies the condition*

$$\sup_{\|w\|\le r+\rho}K_2(t,\tau)\|f_2(\tau,w)\|\le\Phi_{r,\rho}(t,\tau),$$

*where* $\int\limits_0^t\Phi_{r,\rho}(t,\tau)d\tau<\infty$ *for any fixed* $t\in\mathbb{R}_+$.

5. *The inequality*

$$\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)\le\rho$$

*holds for each* $t\in\mathbb{R}_+$.

*Then the equation* $(1),(2)$ *for* $i=2$ *is* $L_p$*-equivalent to the equation* $(1),(2)$ *for* $i=1$ *in the ball* $B_r$.

**Proof.** We shall show that for any function $u_1(t) \in B_r$ $(t \in \mathbb{R}_+)$ the operator $T(u_1, v)$ defined by equality (9) maps the set

$$C(\rho) = \{v \in S(\mathbb{R}_+, X) : \ v(t) \in B_\rho, \ t \in \mathbb{R}_+\}$$

into itself.

Let $u_1(t) \in B_r$ $(t \in \mathbb{R}_+)$ and let $v \in C(\rho)$. Then, making use of (9) we obtain the estimate

$$\begin{aligned}
&\|T(u_1,v)(t)\| \le \|V_2(t,0)(u_1(0)+v(0))\| + \|V_1(t,0)u_1(0)\| + \\
&+ \int_0^t \|V_2(t,\tau)f_2(\tau,u_1(\tau)+v(\tau))\|d\tau + \int_0^t \|V_1(t,\tau)f_1(\tau,u_1(\tau))\|d\tau + \\
&+ \sum_{0<t_n<t} \|V_2(t,t_n^+)h_n^2(u_1(t_n)+v(t_n))\| + \sum_{0<t_n<t} \|V_1(t,t_n^+)h_n^1(u_1(t_n))\| \le \\
&\le K_2(t,0)\|u_1(0)+v(0)\| + K_1(t,0)\|u_1(0)\| + \\
&+ \sup_{\|w\|\le r+\rho} \int_0^t K_2(t,\tau)\|f_2(\tau,w)\|d\tau + \sup_{\|u\|\le r} \int_0^t K_1(t,\tau)\|f_1(\tau,u)\|d\tau + \\
&+ \sup_{\|w\|\le r+\rho} \sum_{0<t_n<t} K_2(t,t_n^+)\|h_n^2(w)\| + \sup_{\|u\|\le r} \sum_{0<t_n<t} K_1(t,t_n^+)\|h_n^1(u)\| \le \\
&\le \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho
\end{aligned}$$

for each $t \in \mathbb{R}_+$.

Let $M = \{m(t) = T(u_1,v)(t) : \ \|v\| \le \rho, \ t \in \mathbb{R}_+\}$.

We shall show the equicontinuity of the functions of the set $M$. Let $t' > t''$ and $t', t'' \in (t_n, t_{n+1}]$. It is easilyseen that.

$$\begin{aligned}
&\|m(t') - m(t'')\| \le \\
&\le \|V_2(t',0)u_2(0) - V_2(t'',0)u_2(0)\| + \|V_1(t',0)u_1(0) - V_1(t'',0)u_1(0)\| + \\
&\quad + \sup_{\|w\|\le r+\rho} \int_0^{t''} \|V_2(t',\tau)f_2(\tau,w) - V_2(t'',\tau)f_2(\tau,w)\|d\tau + \\
&\quad + \sup_{\|u\|\le r} \int_0^{t''} \|V_1(t',\tau)f_1(\tau,u) - V_1(t'',\tau)f_1(\tau,u)\|d\tau + \\
&\quad + \sup_{\|w\|\le r+\rho} \int_{t''}^{t'} K_2(t',\tau)\|f_2(\tau,w)\|d\tau + \sup_{\|u\|\le r} \int_{t''}^{t'} K_1(t',\tau)\|f_1(\tau,u)\|d\tau + \\
&\quad + \sup_{\|w\|\le r+\rho} \sum_{0<t_n<t''} \|V_2(t',t_n^+)h_n^2(w) - V_2(t'',t_n^+)h_n^2(w)\| + \\
&\quad + \sup_{\|u\|\le r} \sum_{0<t_n<t''} \|V_1(t',t_n^+)h_n^1(u) - V_1(t'',t_n^+)h_n^1(u)\|.
\end{aligned}$$

The continuity of functions $V_i(t,\tau)$ $(i=1,2)$ on $(t_n, t_{n+1}]$ and condition 2.1 of Theorem 1 imply the equicontinuity of the set $M$.

It follows from conditions 2.2, 3.2 and (9) that the sections $M(t) = \{m(t) : m \in M\}$ are compact for any $t \in \mathbb{R}_+$. Consequently, Lemma 1 implies the compactness of the set $M$.

Now, we shall show that the operator $T(u_1, v)$ is continuous in $S(\mathbb{R}_+, X)$.

Let the sequence $\{v_k(t)\} \subset C(\rho)$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ to the function $v(t) \in C(\rho)$. Then, for $t \in \mathbb{R}_+$ the sequence $f_2(t, u_1(t) + v_k(t))$ converges to $f_2(t, u_1(t) + v(t))$. Utilizing condition 4 of Theorem 1, we obtaint that the convergent sequence of functions $V_2(t,\tau) f_2(\tau, u_1(\tau) + v_k(\tau))$ is majorized by the integrable function $\Phi_{r,\rho}(t,\tau)$. Therefore, we may pass to the limit in the formula.

$$\begin{aligned}
&T(u_1, v_k)(t) = V_2(t,0)(u_1(0) + v_k(0)) - V_1(t,0)u_1(0) + \\
&+ \int_0^t \{V_2(t,\tau) f_2(\tau, u_1(\tau) + v_k(\tau)) - V_1(t,\tau) f_1(\tau, u_1(\tau))\} d\tau + \\
&+ \sum_{0<t_n<t} \{V_2(t, t_n^+) h_n^2(u_1(t_n) + v_k(t_n)) - V_1(t, t_n^+) h_n^1(u_1(t_n))\}
\end{aligned}$$

Hence, $T(u_1, v_k)(t)$ tends to $T(u_1, v)(t)$ for $t \in \mathbb{R}_+$.

From Lemma 2 it follows that for any $u_1 \in B_r$ the operator $T(u_1, v)$ has a fixed point $v$ in $C(\rho)$ i.e., $v = T(u_1, v)$.

We shall show that this fixed point $v(t)$ lies in $L_p(X)$.

$$\begin{aligned}
&\|v(t)\| \leq K_2(t,0)\|u_1(0) + v(0)\| + K_1(t,0)\|u_1(0)\| + \\
&+ \sup_{\|w\| \leq r+\rho} \int_0^t K_2(t,\tau)\|f_2(\tau, w)\| d\tau + \sup_{\|u\| \leq r} \int_0^t K_1(t,\tau)\|f_1(\tau, u)\| d\tau + \\
&+ \sup_{\|w\| \leq r+\rho} \sum_{0<t_n<t} K_2(t, t_n^+)\|h_n^2(w)\| + \sup_{\|u\| \leq r} \sum_{0<t_n<t} K_1(t, t_n^+)\|h_n^1(u)\| \leq \\
&\leq \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \\
&\|v\|_p \leq \Big(\int_0^\infty |\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t)|^p dt\Big)^{\frac{1}{p}} \leq \\
&\leq \|\chi_{r,\rho}\|_p + \|\psi_{r,\rho}\|_p + \|\varphi_{r,\rho}\|_p
\end{aligned}$$

Hence, this fixed point belongs to the space $S(\mathbb{R}_+, X)$ i.e., equation (1), (2) for $i = 2$ is $L_p$-equivalent to the equation (1), (2) for $i = 1$ in the ball $B_r$.

Theorem 1 is proved. □

We shall illustrate Theorem 1 with an example of the qualitative theory of the nonlinear partial impulse differential equations.

**Example.** In the example we consider two partial impulse differential equations and reduce them to two ordinary impulse differential equtions. For these ordinary impulse differential equations, the conditions of Theorem 1 are fulfilled. Many notations and results for ordinary differential equations are taken from capitals $5-7$ of [4]. The short introduction in the general theory of nonlinear partial impulse differential equations follows [2].

Let $\Omega$ be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $Q=(0,\infty)\times\Omega$ and $\Gamma=(0,\infty)\times\partial\Omega$.

We denote

$$P_n=\{(t_n,x):\ x\in\Omega\},\quad P=\bigcup_{n=1}^{\infty}P_n,$$

$$\Lambda_n=\{(t_n,x):\ x\in\partial\Omega\},\quad \Lambda=\bigcup_{n=1}^{\infty}\Lambda_n.$$

Consider the impulse nonlinear parabolic initial value problems

$$\frac{\partial u_i}{\partial t}=\tilde{A}_i(t,x,D)u_i+\tilde{f}_i(t,x,u_i),\ \ (t,x)\in Q\setminus P \tag{10}$$

$$D^\alpha u_i(t,x)=0,\ \ |\alpha|<m,\ (t,x)\in\Gamma\setminus\Lambda \tag{11}$$

$$u_i(0,x)=v_i(x),\ \ x\in\Omega \tag{12}$$

$$u_i(t_n^+,x)=\tilde{Q}_n^i(u_i(t_n,x))+\tilde{h}_n^i(u_i(t_n,x)),\ \ x\in\overline{\Omega},\ n=1,2,..., \tag{13}$$

where

$$\tilde{A}_i(t,x,D)=\sum_{|\alpha|\le 2m}a_\alpha^i(t,x)D^\alpha,$$

$\tilde{Q}_n^i : D(\tilde{Q}_n^i)\to D(\tilde{A}_i(t_n,x,D))$ $(n=1,2,...;i=1,2)$ are linear operators, $\tilde{f}_i(.,.,.):\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ and $\tilde{h}_n^i:\mathbb{R}\to\mathbb{R}$ are continuous functions.

Let $X=L_p(\Omega,\mathbb{R})$ $(1<p<\infty)$, where

$$L_p(\Omega,\mathbb{R})=\{v:\Omega\to\mathbb{R};\ \int_\Omega|v(x)|^pdx<\infty\}$$

with norm $|v|_p = (\int\limits_{\Omega} |v(x)|^p dx)^{\frac{1}{p}}$.

With the family $\tilde{A}_i(t,x,D),\ t \in \mathbb{R}_+,\ (i=1,2)$ of strongly elliptic operators we associate a family of linear operators $A_i(t),\ t \in \mathbb{R}_+,\ (n=1,2)$ acting in $X$ by

$$A_i(t)u_i = \tilde{A}_i(t,x,D)u_i, \ \text{ for } u_i \in D.$$

This is done as follows $D = D(A_i(t)) = W^{2m,p}(\Omega) \bigcap W_0^{m,p}(\Omega),\ (i{=}1,2;\ t \in \mathbb{R}_+)$.

Let $v_i \in X$. We set

$$f_i(t,u_i)(x) = \tilde{f}_i(t,x,u_i(t,x)),\ \ u_i \in X,\ t \in \mathbb{R}_+,\ x \in \overline{\Omega}\ (i=1,2),$$
$$Q_n^i(u_i(t_n))(x) = \tilde{Q}_n^i(u_i(t_n,x)),\ \ h_n^i(u_i(t_n))(x) = \tilde{h}_n^i(u_i(t_n,x)),$$

where $Q_n^i : D(Q_n^i) \to D \ \ (D(Q_n^i) \subset X$ lie dense in $X \ \ (i=1,2))$ are linear operators, $f_n^i : \mathbb{R}_+ \times X \to X$ and $h_n^i : X \to X$ are continuous functions.

We shall prove the $L_p$-equivalence between the equations $(1),(2)$ $(i=1,2)$.

Let $U_i(t,s)$ $(i=1,2)$ are the Cauchy operators of the equations

$$\frac{du_i}{dt} = A_i(t)u_i$$

Sufficient conditions for the validity of the estimates

$$|U_i(t,s)|_{p\to p} \leq C_i e^{-k_i(t-s)} \ \ (0 \leq s \leq t;\ C_i, k_i > 0 \text{ constants, } i=1,2)$$

are given in [4].

We shall consider the concrete case when $t_n = n$ $(n=1,2,...)$,

$$\tilde{f}_1(t,x,u_1) = e^{\gamma_1 t}\tfrac{u_1^2(t,x)}{1+u_1^2(t,x)},\ \ \tilde{f}_2(t,x,u_2) = e^{\gamma_2 t}\sin u_2(t,x),$$

$$\tilde{Q}_n^1\xi_1 = \tfrac{k_1 n}{C_1(1+n^2)e^{C_1+k_1}}\xi_1,\ \ \tilde{Q}_n^2\xi_2 = \tfrac{k_2 n}{C_2(1+n^2)e^{C_2+k_2}}\xi_2,$$

$$\tilde{h}_n^1(u_1(t_n,x)) = e^{\alpha_1 n}.2^{-u_1(t_n,x)},\ \ \tilde{h}_n^2(u_2(t_n,x)) = e^{\alpha_2 n}\tfrac{1}{1+u_2^2(t_n,x)}$$

where $-1 < \gamma_i + k_i < 0$ and $\alpha_i + k_i < \ln\frac{1}{2}$ $(i=1,2)$.
Then

$$f_1(t,u_1) = e^{\gamma_1 t}\tfrac{u_1^2(t)}{1+u_1^2(t)},\ \ f_2(t,u_2) = e^{\gamma_2 t}\sin u_2(t),$$

$$Q_n^1\xi_1 = \tfrac{k_1 n}{C_1(1+n^2)e^{C_1+k_1}}\xi_1,\ \ \tilde{Q}_n^2\xi_2 = \tfrac{k_2 n}{C_2(1+n^2)e^{C_2+k_2}}\xi_2,$$

$$h_n^1(u_1(t_n)) = e^{\alpha_1 n}.2^{-u_1(t_n)},\ \ h_n^2(u_2(t_n)) = e^{\alpha_2 n}\tfrac{1}{1+u_2^2(t_n)}\,.$$

Let $V_i(t,s)$ $(i=1,2;\ 0\leq s\leq t)$ are the Cauchy operators of the linear impulse equations

$$\frac{du_i}{dt}=A_i(t)u_i \text{ for } t\neq t_n$$
$$u_i(t_n^+)=Q_n^i(u_i(t_n)) \text{ for } n=1,2,...$$

Then for $0<s\leq k<n<t,\ \xi\in D$ the following estimates are valid

$$|V_1(t,s)\xi|_p=|U_1(t,t_n)Q_n^1...Q_k^1U_1(t_k,s)\xi|_p\leq$$
$$\leq C_1e^{-k_1(t-n)}\frac{k_1n}{C_1(1+n^2)e^{C_1+k_1}}\cdots\frac{k_1k}{C_1(1+k^2)e^{C_1+k_1}}C_1e^{-k_1(k-s)}|\xi|_p\leq$$
$$\leq\frac{C_1}{e^{C_1(n-k+1)}}\frac{k_1^{n-k}}{e^{k_1(n-k+1)}}k_1ne^{-k_1(t-s)}|\xi|_p\leq k_1te^{-k_1(t-s)}|\xi|_p.$$

Similarly

$$|V_2(t,s)\xi|_p\leq k_2te^{-k_2(t-s)}|\xi|_p.$$

We set

$$k_i(t,s)=k_ite^{-k_i(t-s)}\ \ (i=1,2)$$

Let $r>0$ and

$$\rho>\frac{2}{e-1}(r+2(\mu(\Omega))^{\frac{1}{p}}) \tag{14}$$

We shall show that the conditions of Theorem 1 are fulfilled.

For any $\xi\in B_r$, $\eta\in B_{r+\rho}$, $t\in\mathbb{R}_+$ we obtain

$$K_1(t,0)|\xi|_p+K_2(t,0)|\eta|_p=k_1te^{-k_1t}|\xi|_p+k_2te^{-k_2t}|\eta|_p\leq$$
$$\leq k_1te^{-k_1t}r+k_2te^{-k_2t}(r+\rho).$$

Let us set

$$\chi_{r,\rho}(t)=k_1te^{-k_1t}r+k_2te^{-k_2t}(r+\rho).$$

We shall show that condition 2.1 of Theorem 1 is fulfilled.

$$\sup_{|u|_p\leq r}\int_0^t K_1(t,\tau)|f_1(\tau,u)|_pd\tau+\sup_{|w|_p\leq r+\rho}\int_0^t K_2(t,\tau)|f_2(\tau,w)|_pd\tau=$$
$$=\sup_{|u|_p\leq r}\int_0^t k_1te^{-k_1(t-\tau)}e^{\gamma_1\tau}\left|\frac{u^2(\tau)}{1+u^2(\tau)}\right|_pd\tau+$$
$$+\sup_{|w|_p\leq r+\rho}\int_0^t k_2te^{-k_2(t-\tau)}e^{\gamma_2\tau}|\sin w(\tau)|_pd\tau\leq$$

$$\leq k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\int\limits_0^t e^{(k_1+\gamma_1)\tau}d\tau+$$
$$+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\int\limits_0^t e^{(k_2+\gamma_2)\tau}d\tau \leq$$
$$\leq k_1te^{-k_1t}\tfrac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_1+\gamma_1)}+k_2te^{-k_2t}\tfrac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_2+\gamma_2)}$$

Let us set

$$\psi_{r,\rho}(t)=k_1te^{-k_1t}\frac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_1+\gamma_1)}+k_2te^{-k_2t}\frac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_2+\gamma_2)}.$$

We shall prove condition 3.1 of Theorem 1.

$$\sup_{|u|_p\leq r}\sum_{0<n<t}K_1(t,t_n^+)|h_n^1(u(t_n))|_p+\sup_{|w|_p\leq r+\rho}\sum_{0<n<t}K_2(t,t_n^+)|h_n^2(w(t_n))|_p=$$
$$=\sup_{|u|_p\leq r}\sum_{0<n<t}k_1te^{-k_1(t-n)}e^{\alpha_1n}|2^{-u(t_n)}|_p+$$
$$+\sup_{|w|_p\leq r+\rho}\sum_{0<n<t}k_2te^{-k_2(t-n)}e^{\alpha_2n}|\tfrac{1}{1+w^2(t_n)}|_p\leq$$
$$\leq k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\sum_{0<n<t}e^{(k_1+\alpha_1)n}+$$
$$+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\sum_{0<n<t}e^{(k_2+\alpha_2)n}\leq$$
$$\leq k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}}+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}$$

Set

$$\varphi_{r,\rho}(t)=k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\frac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}}+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\frac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}.$$

It is not hard to check if the functions $\chi_{r,\rho}(t)$, $\psi_{r,\rho}(t)$ and $\varphi_{r,\rho}(t)$ lie in the space $L_p(\mathbb{R}_+)$.

Condition 4 of Theorem 1 is fulfilled with

$$\Phi_{r,\rho}(t,\tau)=k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}e^{(k_2+\gamma_2)\tau}\in L_1(\mathbb{R}_+)$$

for any fixed $t\in\mathbb{R}_+$.

We shall show that condition 5 of Theorem 1 holds

$$\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)=$$
$$=k_1te^{-k_1t}(r-(\mu(\Omega))^{\frac{1}{p}}\tfrac{1}{k_1+\gamma_1}+(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}})+$$
$$+k_2te^{-k_2t}(r+\rho-(\mu(\Omega))^{\frac{1}{p}}\tfrac{1}{k_2+\gamma_2}+(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}).$$

From condition (14) we obtain

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \leq \rho \text{ for each } t \in \mathbb{R}_+.$$

By means of a compactness criterion from [3] we shall prove condition 2.2. Set

$$M(t) = \{m(t) = \int_0^t V_2(t,\tau) f_2(\tau, u_1(\tau) + v(\tau)) d\tau : \ |v|_p \leq \rho\}$$

is a compact subset of $X$ for any fixed $t$.

Indeed,

$$|m(t)(x)| \leq k_2 t e^{-k_2 t} \int_0^t e^{(k_2+\gamma_2)\tau} |\sin(v(\tau)(x) + u_1(\tau)(x))| d\tau \leq$$
$$\leq \int_0^t e^{(k_2+\gamma_2)\tau} d\tau = \frac{1}{k_2+\gamma_2}(e^{(k_2+\gamma_2)t} - 1), \text{ i.e.}$$
$$(\int_\Omega |m(t)(x)|^p dx)^{\frac{1}{p}} \leq \frac{1}{k_2+\gamma_2}(e^{(k_2+\gamma_2)t} - 1)(\mu(\Omega))^{\frac{1}{p}}$$

and hence $|m(t)(x)|_p \leq N$ ($N$-constant).

We show that

$$|m(t)(x+h) - m(t)(x)|_p \to 0 \ \ (h \to 0).$$

This follows from the relations below

$$|m(t)(x+h) - m(t)(x)| \leq$$
$$\leq \int_0^t e^{(k_2+\gamma_2)\tau} |\sin(v(\tau)(x+h) + u_1(\tau)(x+h)) - \sin(v(\tau)(x) + u_1(\tau)(x))| d\tau \leq$$
$$\leq \int_0^t e^{(k_2+\gamma_2)\tau} |v(\tau)(x+h) - v(\tau)(x)| d\tau + \int_0^t e^{(k_2+\gamma_2)\tau} |u(\tau)(x+h) - u(\tau)(x)| d\tau$$

In a similar way, we show the validily of condition 3.2.

The conditions of Theorem 1 are fulfilled and hence the ordinary equations $(1), (2)$ $(i = 1, 2)$ are in $B_r$ $L_p$-equivalent. Hence, every solution $u_1(t,x)$ of $(10) - (13)$ $(i = 1)$ induces a solution $u_2(t,x)$ of $(10) - (13)$ $(i = 2)$ such that the function $\alpha_1(t) = |u_1(t,x) - u_2(t,x)|$ lies in $L_p(\mathbb{R}_+)$ for any $x \in \Omega$ .

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# $L_p$-ЕКВИВАЛЕНТНОСТ МЕЖДУ ДВЕ НЕЛИНЕЙНИ ИМПУЛСНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С НЕОГРАНИЧЕНИ ЛИНЕЙНИ ЧАСТИ И ТЯХНОТО ПРИЛОЖЕНИЕ ЗА ЧАСТНИ ИМПУЛСНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ

Атанаска Георгиева, Степан Костадинов

**Резюме**. С помощта на теоремата на Шаудер–Тихонов за неподвижната точка е доказана $L_p$-еквивалентност между две импулсни диференциални уравнения с неограничени линейни части. Даден е пример от теорията на частните импулсни диференциални уравнения от параболичен тип.