On Weak Uniform Normal Structure in Weighted Orlicz Sequence Spaces[1]

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ABSTRACT: It is proved that a weighted Orlicz sequence space $\ell_M(w)$, equipped with Luxemburg or Amemiya norm has weak uniform normal structure iff $\ell_M(w) \cong h_M(w)$ for wide class of weight sequences $w = \{w_n\}_{n=1}^{\infty}$. An example is constructed, where M has not Δ_2–condition but by choosing a suitable weight sequence $\lim_{n\to\infty} w_n = \infty$ we get that $\ell_M(w)$ has weak uniform normal structure.

Keywords: weighted Orlicz sequence space, Luxemburg norm, Amemiya norm, weak uniform normal structure, weakly convergent sequence coefficient.

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1 Introduction

The weakly convergent sequence coefficient $WCS(X)$ of a Banach space X was introduced by Bynum in [4]. The connections of the coefficient $WCS(X)$ with some geometric parameters were investigated in [4], [28], [16], [20]. The notation of normal structure was introduced by Brodskii and Milman in [3]. It is well known that a Banach space with normal structure has the weak fixed point property [1], [4], [9], [20]. A reflexive Banach space X with $WCS(X) > 1$ has normal structure [4] and consequently it has the weak fixed point property, which means that each nonexpansive mapping of nonempty convex weakly compact set in X has fixed point [15].

Banach space with $WCS(X) > 1$ is said to have weak uniform normal structure or some authors prefer to say that X is a Bynum's space.

If X is a monotone complete Köthe sequence space and $WCS(X) > 1$ then X is order continuous [7]. It is shown in [24] that Bynum's condition implies strong subsequential property (P) which in turn implies subsequential property (P). In the same article it is proved that for an Asplund spaces Bynum's condition is equivalent to subsequential property (P).

If $WCS(X) > 1$ then X has weakly normal structure and thus any nonexpansive mapping defined on convex weakly compact subsets of X has a fixed point [2].

A large class of Banach spaces verify Bynum's condition. Let us just mention the spaces with uniform normal structure and uniformly convex spaces. In [6] it is proved that an Orlicz sequence spaces equipped with Luxemburg or Amemiya norm has weak uniform normal structure iff M has Δ_2–condition.

T. Kim and E. Kim have found a sufficient condition for asymptotically regular maps $T : C \to C$ to have iterative fixed point for Banach spaces X with $WCS(X) > 1$, where C is a nonempty closed convex subset of X [14].

Let us mention that the weakly convergent sequence constant WCS depends on the norm, i.e. it can change in equivalent renormings. The exact value of WCS is found for some Banach spaces, equipped with the usual norms. For $p \geq 1$ $WCS(\ell_p) = 2^{1/p}$ [4] and $WCS(L_p(\Omega)) = \min\{2^{1/p}, 2^{1-1/p}\}$ [22]. For a Hilbert space H it is well known that

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$WCS(H) = 2^{1/2}$ and $WCS(c_0) = 1$. A formula for calculating the WCS coefficient of reflexive Orlicz and Musielak–Orlicz sequence spaces equipped with Luxemburg or Amemiya norm is found in [6] and [27], respectively.

2 Preliminaries

We use the standard Banach space terminology from [17]. Let X be a real Banach space, S_X be the unit sphere of X. Let ℓ^0 stand for the space of all real sequences i.e. $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$, $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ is the set of the real numbers.

For a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of X, we define

$$A(\{x^{(n)}\}) = \lim_{n\to\infty} \sup\{\|x^{(i)} - x^{(j)}\| : i, j \geq n, i \neq j\}$$

and

$$A_1(\{x^{(n)}\}) = \lim_{n\to\infty} \inf\{\|x^{(i)} - x^{(j)}\| : i, j \geq n, i \neq j\}.$$

Definition 2.1 *The weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined as follows:*

$$\begin{aligned} WCS(X) &= \sup\{k : \textit{for each weakly convergent sequence } \{x^{(n)}\}_{n=1}^{\infty}, \textit{ there exists} \\ &\qquad \textit{some } y \in \operatorname{co}(\{x^{(n)}\}_{n=1}^{\infty}) \textit{ such that } k \limsup_{n\to\infty} \|x^{(n)} - y\| \leq A(\{x^{(n)}\})\}, \end{aligned}$$

where $\operatorname{co}(\{x^{(n)}\}_{n=1}^{\infty})$ denotes the convex hull of the elements of $\{x^{(n)}\}_{n=1}^{\infty}$.

It is easy to see that $1 \leq WCS(X) \leq 2$.

Recall that a Banach space has Schur property if every weakly null sequence is norm null. We will assume in the sequel that the Banach spaces, we investigate are not Schur spaces. Thus there exists a weakly null sequence $\{x^{(n)}\}_{n=1}^{\infty} \in X$, which is not norm null. We will use the notation $x^{(n)} \xrightarrow{w} 0$ to indicate that $\{x^{(n)}\}_{n=1}^{\infty}$ converges weakly to zero.

Definition 2.2 *[28] A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be asymptotic equidistant sequence if $A(\{x^{(n)}\}) = A_1(\{x^{(n)}\})$.*

The result that

$$WCS(X) = \inf\{A(\{x^{(n)}\}) : \{x^{(n)}\}_{n=1}^{\infty} \subset S_X, A(\{x^{(n)}\}) = A_1(\{x^{(n)}\}), x^{(n)} \xrightarrow{w} 0\}$$

is obtained in [28].

Definition 2.3 *A Banach space X is said to have weak uniform normal structure if $WCS(X) > 1$ [11].*

Definition 2.4 *A Banach space $(X, \|\cdot\|)$ is said to be Köthe sequence space if X is a subspace of ℓ^0 such that*

i) If $x \in \ell^0$, $y \in X$ and $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \leq \|y\|$;

ii) There exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.

A sequence $\{v_i\}_{i=1}^{\infty}$ in a Banach space X is called Schauder basis of X (or basis for short) if for each $x \in X$ there exists an unique sequence $\{a_i\}_{i=1}^{\infty}$ of scalars such that $x = \sum_{i=1}^{\infty} a_i v_i$. If $\{v_i\}_{i=1}^{\infty}$ is a basis in X such that the series $\sum_{i=1}^{\infty} a_i v_i$ converges whenever $\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} a_i v_i \right\| < \infty$, then it is called a boundedly complete basis of X. A sequence of non zero vectors $\{x^{(n)}\}_{n=1}^{\infty}$ of the form $\sum_{i=p_n+1}^{p_{n+1}} a_i v_i$, with $\{a_i\}_{i=1}^{\infty}$ scalars and $0 = p_1 < p_2 < p_3 \ldots$ an increasing sequence of integers is called a block basic sequence or block basis of $\{v_i\}_{i=1}^{\infty}$ for short. By $\{e_i\}_{i=1}^{\infty}$ we denote the unit vectors.

The main tool in this note will be the next theorem:

Theorem 1 *([6])Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$–boundedly complete basis. Then*

$$WCS(X) = \inf \left\{ A(\{x^{(n)}\}) : x^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} x_n(i) e_i \in S_X, x_n \xrightarrow{w} 0, 0 = p_1 < p_2 < p_3 \ldots \right\}.$$

Let us recall that an Orlicz function M is an even, continuous, nondecreasing convex function such that $M(0) = 0$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ of Orlicz functions is called a Musielak–Orlicz function or a MO function in short.

The MO sequence space ℓ_Φ, generated by a MO function Φ is the set of all real sequences $\{x_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The space ℓ_Φ is a Banach space if endowed with the Luxemburg's norm:

$$\|x\|_\Phi = \inf \left\{ r > 0 : \sum_{i=1}^{\infty} \Phi_i(x_i/r) \le 1 \right\}$$

or Amemiya's norm:

$$|x|_\Phi = \inf \left\{ \frac{1}{k} \left(1 + \sum_{i=1}^{\infty} \Phi_i(k x_i) \right) : k > 0 \right\}.$$

These norms are connected by the inequalities

$$\|\cdot\|_\Phi \le |\cdot|_\Phi \le 2\|\cdot\|_\Phi. \tag{1}$$

Throughout this note we always denote by M an Orlicz function and by Φ a MO–function.

If the MO function Φ consists of one and the same Orlicz function M we get the Orlicz sequence space denoted by ℓ_M.

A weight sequence $w = \{w_i\}_{i=1}^{\infty}$ is a sequence of positive reals. We will distinguish two classes of weighted sequences Λ_∞ and Λ. The weight sequence $w = \{w_i\}_{i=1}^{\infty}$ is from the class Λ_∞ if it is nondecreasing sequence with $\lim_{i\to\infty} w_i = \infty$. Following [10] we say that $w = \{w_i\}_{i=1}^{\infty}$ is from the class Λ if there exists a subsequence $w = \{w_{i_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} w_{i_k} = 0$ and $\sum_{k=1}^{\infty} w_{i_k} = \infty$.

A weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M and a weight sequence w is the MO sequence space ℓ_Φ, where $\Phi_i(t) = w_i M(t)$.

Weighted Orlicz sequence spaces were investigated for example in [8], [21] and most recently in [13]. Let us mention that if the weight sequence is from the class Λ, then a lot of the properties of the space $\ell_M(w)$ depend only on the generating Orlicz function M, which is in contrast with the results when $w \notin \Lambda$ [10], [23], [18].

It is well known that the Orlicz and the weighted Orlicz sequence spaces equipped with Luxemburg or Amemiya norms are Köthe sequence spaces.

For simplicity of notations we will use $\widetilde{\Phi}(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)$ and $\widetilde{M}_w(x) = \sum_{i=1}^{\infty} w_i M(x_i)$.

An extensive study of Orlicz and MO spaces can be found in [17], [25].

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x \in \ell_\Phi$, such that $\widetilde{\Phi}(\lambda x_i) < \infty$ for every $\lambda > 0$ and by $h_M(w)$ the subspace of $\ell_M(w)$ such that $\widetilde{M}_w(\lambda x) < \infty$ for every $\lambda > 0$.

The unit vectors $\{e_i\}_{i=1}^{\infty}$ is a boundedly complete basis in h_Φ, equipped with the Luxemburg or Amemiya norm.

We say that M has Δ_2–condition if there exist $C > 1$ and $t_0 > 0$ such that $M(2t) \le CM(t)$ for every $t \in (0, t_0]$.

If $w \in \Lambda$ then the spaces $\ell_M(w)$ and $h_M(w)$ coincide iff $M \in \Delta_2$. The proof is similar to that done in ([17] Proposition 4.a.4).

To every Orlicz function M the following number is associated (see [17], p. 143)

$$\beta_M = \inf\{p : \inf\{M(uv)/u^p M(v) : u, v \in (0, 1]\} > 0\}.$$

An Orlicz function M satisfies the Δ_2–condition iff $\beta_M < \infty$, which implies of course $M(uv) \ge u^q M(v)$, $u, v \in [0, 1]$ for some $q \ge \beta_M$ (see [17] p.140).

Definition 2.5 *We say that the MO function Φ satisfies the δ_2–condition if there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that for every $n \in \mathbb{N}$*

$$\Phi_n(2t) \le K\Phi_n(t) + c_n, \tag{2}$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide iff Φ has δ_2–condition.

We say that the MO function Φ satisfies the uniform δ_2–condition if it satisfies (2) for every $t \in [0, t_0]$ for some $t_0 > 0$ with $c_n = 0$ for every $n \in \mathbb{N}$.

Recall that given MO functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ, that is there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \le \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \le \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$, [12] and [19].

If $\Phi_i(1) = 1$ for every $i \in \mathbb{N}$ then the unit vectors $\{e_i\}_{i=1}^{\infty}$ is a normalized boundedly complete basis in h_Φ. If $\Phi_i(1) \neq 1$, let a_i be the solution of the equation $\Phi_i(a_i) = 1$, $i \in \mathbb{N}$ then ℓ_Φ, equipped with the Luxemburg or Amemiya norm, is isometric to ℓ_φ, where the MO function φ is defined by the sequence $\varphi_i(t) = \dfrac{\Phi_i(a_i t)}{\Phi_i(a_i)}$. Therefore the sequence $v_i = a_i e_i$, $i \in \mathbb{N}$ is a normalized boundedly complete basis in h_Φ. The unit vector basis $\{e_i\}_{i=1}^{\infty}$ is a boundedly complete basis in ℓ_φ iff $\ell_\varphi \cong h_\varphi$. The weighted Orlicz sequence space $\ell_M(w)$ is isometric to ℓ_φ, where the MO function φ is defined by

$$\varphi_i(t) = \frac{M(a_i t)}{M(a_i)}, \quad a_i = M^{-1}(1/w_i). \tag{3}$$

Hence $\ell_M(w) \cong h_M(w)$ iff $\ell_\varphi \cong h_\varphi$ and $y = \sum_{i=1}^{\infty} x_i e_i \in h_\varphi$ iff $x = \sum_{i=1}^{\infty} a_i x_i e_i \in h_M(w)$. As the spaces $\ell_M(w)$ and ℓ_φ are isometric then $WCS(\ell_M(w)) = WCS(\ell_\varphi)$ and if $\ell_M(w) \cong h_M(w)$ according to Theorem 1 there exist weakly null sequences $\{x^{(n)}\}_{n=1}^{\infty}$, $\{y^{(n)}\}_{n=1}^{\infty}$, $x^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i^{(n)} e_i \in S_{\ell_M(w)}$ and $y^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} x_i^{(n)} e_i \in S_{\ell_\varphi}$, such that

$$WCS(\ell_M(w)) - \varepsilon \le A(\{x^{(n)}\}) = A(\{y^{(n)}\}) \le WCS(\ell_M(w)) + \varepsilon.$$

Definition 2.6 *([26]) A MO function Φ is said to satisfy the uniform Δ_2–condition if there exist $q \ge 1$ and $i_0 \in \mathbb{N}$, such that for all $t \in (0,1]$ and $i \ge i_0$ we have*

$$\frac{t p_i(t)}{\Phi_i(t)} \le q, \tag{4}$$

where p_i is the right derivative of Φ_i.

Let $\ell_M(w) \cong h_M(w)$ then $\ell_\varphi \cong h_\varphi$, where $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ is defined in (3). Now following the construction done by Woo ([26] Theorem 3.5) there exist $m \in \mathbb{N}$ and a sequence $\{x_i\}_{i=1}^{\infty}$, such that $\sum_{i=1}^{\infty} \varphi_i(x_i) < \infty$ and

$$x p_i'(x) \le 2^m \varphi_i(x) \ \text{ for every } \ x \in [x_i, 1], \tag{5}$$

where p_i' is the right derivative of φ_i. After choosing y_i to be the solution of the equation $\varphi_i(y_i) = \varphi_i(x_i) + 2^{-i}$, $i \in \mathbb{N}$ we get that

1) $\sum_{i=1}^{\infty} \varphi_i(y_i) < \infty$;

2) the MO function $\Psi = \{\Psi_i\}_{i=1}^{\infty}$, defined by

$$\Psi_i(t) = \begin{cases} \varphi_i(t), & t \ge y_i \\ \dfrac{t\varphi_i(y_i)}{y_i}, & t \le y_i \end{cases} \tag{6}$$

has the uniform Δ_2–condition with $i_0 = 1$.

By the inequalities $\varphi_i(t) \le \Psi_i(t)$ and $\Psi_i(t) \le \varphi_i(t) + \varphi_i(y_i)$ for every $t \ge 0$ it follows that Ψ is equivalent to φ and therefore $\ell_M(w) \cong \ell_\Psi$.

Throughout this note we will denote by Ψ and $y = \{y_i\}_{i=1}^{\infty}$ the MO function and the sequence defined in (6) and by $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ the MO function defined in (3).

3 Main result

Theorem 2 *Let M be an Orlicz function such that $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $w = \{w_i\}_{i=1}^{\infty}$ be a weight sequence either form the class Λ or Λ_∞. Then the weighted Orlicz sequence space $\ell_M(w)$ endowed with the Luxemburg or Amemiya norm has weak uniform normal structure iff $\ell_M(w) \cong h_M(w)$.*

4 Auxiliary results

We need the following results:

Theorem 3 *[5] A sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in $\ell_M(w)$ is weakly null iff*

(a) $\lim_{n\to\infty} x_i^{(n)} = 0$ for all $i \in \mathbb{N}$;

(b) $\lim_{\lambda\to 0} \sup_{n\in\mathbb{N}} \frac{\widetilde{M}_w(\lambda x^{(n)})}{\lambda} = 0$;

(c) For any subsequence $\{x^{(n_k)}\}_{k=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ holds: $\lim_{m\to\infty} \theta(\min_{k\le m} |x^{(n_k)}|) = 0$, where $\theta(x) = \inf\left\{\lambda > 0 : \widetilde{M}(x/\lambda) < \infty\right\}$ and $\min_{k\le m} |x^{(n_k)}| = \left\{\min_{k\le m} |x_i^{(n_k)}|\right\}_{i=1}^{\infty}$.

Lemma 4.1 *Let $\ell_M(w)$ be a weighted Orlicz sequence space, generated by an Orlicz function M such that $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and a weight sequence $w \in \Lambda$. Then any block basic sequence $x_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha e_i$ with constant coefficients, such that $\sup_{n\in\mathbb{N}} \sum_{i=p_n+1}^{p_{n+1}} w_i \le K < \infty$ is a weakly null sequence.*

Proof: We need to check the conditions in Theorem 3.

Obviously for any block basic sequence holds $\lim_{n\to\infty} x_i^{(n)} = 0$ for every $i \in \mathbb{N}$ and by $\min_{k\le m} |x^{(n_k)}| = 0$ it follows that $\theta(\min_{k\le m} |x^{(n_k)}|) = 0$ for every $m \in \mathbb{N}$.

By $\lim_{t\to 0} \frac{M(t)}{t} = 0$ it follows that for every $\varepsilon > 0$ there exists $t_0 > 0$ such that for every $t \in (0, t_0]$ holds $\frac{M(t)}{t} < \frac{\varepsilon}{\alpha K}$. Then for any $\lambda > 0$ such that $0 < \alpha\lambda \le t_0$ the inequalities hold

$$\frac{\widetilde{M}_w(\lambda x^{(n)})}{\lambda} = \sum_{i=p_n+1}^{p_{n+1}} w_i \frac{M(\lambda\alpha)}{\lambda} = \sum_{i=p_n+1}^{p_{n+1}} w_i \frac{M(\lambda\alpha)}{\lambda\alpha}\alpha < \frac{\varepsilon}{\alpha K} \sum_{i=p_n+1}^{p_{n+1}} w_i \alpha \le \varepsilon.$$

Therefore

$$\lim_{\lambda\to 0} \sup_{n\in\mathbb{N}} \frac{\widetilde{M}_w(\lambda x^{(n)})}{\lambda} = 0.$$

$\square$

Remark: Lemma 4.1 holds true for any block basic sequence $x^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i e_i$ with $\sup_{i\in\mathbb{N}} |\alpha_i| \le \alpha < \infty$.

Lemma 4.2 *Let $\ell_M(w)$ be a weighted Orlicz sequence space, generated by an Orlicz function M such that $\lim\limits_{t\to 0}\dfrac{M(t)}{t}=0$ and a weight sequence $w\in\Lambda_\infty$. Then any block basic sequence $y^{(n)}=\sum\limits_{i=p_n+1}^{p_{n+1}} y_i^{(n)}e_i$, such that $\sup\limits_{n\in\mathbb{N}}\sum\limits_{i=p_n+1}^{p_{n+1}} w_iM(y_i^{(n)})\le C<\infty$ is a weakly null sequence.*

Proof: Conditions (a) and (c) in Theorem 3 are fulfilled as like as in Lemma 4.1.

Observe that by the fact that $\dfrac{M(tu)}{tM(u)}$ is an increasing function of $u\in(0,+\infty)$ [17] it follows that for any $\varepsilon>0$ there exists $t_0>0$ so that for every $t\in(0,t_0]$ and $u\in(0,1]$ holds

$$\frac{M(tu)}{tM(u)}\le\frac{M(t)}{t}<\varepsilon/C. \tag{7}$$

Consequently by (7) and the fact that $|y_n(i)|\le 1$ for $n\in\mathbb{N}$ and $i=p_n+1,\dots,p_{n+1}$ we get

$$\begin{aligned}\frac{\widetilde{M}_w(ty^{(n)})}{t} &= \sum_{i=p_n+1}^{p_{n+1}} w_i\frac{M(ty_i^{(n)})}{t}=\sum_{i=p_n+1}^{p_{n+1}} w_i\frac{M(ty_i^{(n)})}{tM(y_i^{(n)})}M(y_i^{(n)})\\ &< \frac{\varepsilon}{C}\sum_{i=p_n+1}^{p_{n+1}} w_iM(y_i^{(n)})\le\varepsilon.\end{aligned}$$

□

In the case when we want to prove $WCS(\ell_M(w))=1$ for $w\in\Lambda$, WLOG we may assume that $\lim_{n\to\infty}w_n=0$ and $\sum_{n=1}^\infty w_n=\infty$. If not we may consider the subspace $\ell_M(\{w_{n_k}\})\hookrightarrow\ell_M(w)$ and if we show that $WCS(\ell_M(\{w_{n_k}\}))=1$ by the inequality $WCS(\ell_M(\{w_{n_k}\}))\ge WCS(\ell_M(w))$ it will follow $WCS(\ell_M(w))=1$.

Lemma 4.3 *Let $(\ell_M(w),|\cdot|)$ be a weighted Orlicz sequence space, generated by an Orlicz function $M\notin\Delta_2$ and a weight sequence $w\in\Lambda$. Then for any $\varepsilon>0$ there exists a sequence $\{y^{(n)}\}_{n=1}^\infty$, $|y^{(n)}|=1$ such that*

$$A(\{y^{(n)}\})\le(1+\varepsilon)(1+4M(2\varepsilon)+4\varepsilon).$$

Proof: Let $w\in\Lambda$ and $\ell_M(w)$ be equipped with the Amemiya's norm $|\cdot|$. By $M\notin\Delta_2$ it follows that for any $\varepsilon>0$ there is $u>0$ such that $u<\varepsilon$ and $\varepsilon M((1+\varepsilon)u)>M(u)$. Setting $v=(1+\varepsilon)u$, we get the inequality $M\left(\dfrac{v}{1+\varepsilon}\right)<\varepsilon M(v)$.

Since $v<2\varepsilon$ we can find a positive integer m and $\delta>0$ such that

$$1-M(2\varepsilon)<mM(v)\le 1\quad\text{and}\quad(m+\delta)M(v)<2.$$

Take $c\ge 0$ satisfying $mM(v)+M(c)=1$. Then $M(c)<M(2\varepsilon)$.

Choose two sequences of naturals $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ such that

$$p_1<q_1<p_2<q_2<\ldots<p_n<q_n<p_{n+1}<\ldots$$

and

$$1 \leq \sum_{i=p_n+1}^{q_n} w_i \leq 1+\delta \quad m \leq \sum_{i=q_n+1}^{p_{n+1}} w_i \leq m+\delta.$$

Put

$$x^{(n)} = \sum_{i=p_n+1}^{q_n} ce_i + \sum_{i=q_n+1}^{p_{n+1}} ve_i. \tag{8}$$

By

$$\sum_{i=p_n+1}^{q_n} w_i M(c) + \sum_{i=q_n+1}^{p_{n+1}} w_i M(v) \geq M(c) + mM(v) = 1$$

it follows that $|x^{(n)}|_M \geq 1$. Put $z^{(n)} = \dfrac{x^{(n)}}{|x^{(n)}|_M}$. By Lemma 4.1 the sequence $\{z^{(n)}\}_{n=1}^{\infty}$ is normalized weakly null sequence. Finally we estimate $A(\{z^{(n)}\})$:

$$\begin{aligned}
\left|\frac{z^{(n)}-z^{(k)}}{1+\varepsilon}\right|_M &= \left|\frac{z^{(n)}+z^{(k)}}{1+\varepsilon}\right|_M = \left|\frac{1}{1+\varepsilon}\left(\frac{x^{(n)}}{|x^{(n)}|_M}+\frac{x^{(k)}}{|x^{(k)}|_M}\right)\right|_M \\
&\leq \left|\frac{x^{(n)}+x^{(k)}}{1+\varepsilon}\right|_M \leq 1 + \sum_{i=p_n+1}^{q_n} w_i M\left(\frac{x_i^{(n)}}{1+\varepsilon}\right) + \sum_{i=q_n+1}^{p_{n+1}} w_i M\left(\frac{x_i^{(n)}}{1+\varepsilon}\right) \\
&+ \sum_{i=p_k+1}^{q_k} w_i M\left(\frac{x_i^{(k)}}{1+\varepsilon}\right) + \sum_{i=q_k+1}^{p_{k+1}} w_i M\left(\frac{x_i^{(k)}}{1+\varepsilon}\right) \\
&= 1 + \left(\sum_{i=p_n+1}^{q_n} w_i + \sum_{i=p_k+1}^{q_k} w_i\right) M\left(\frac{c}{1+\varepsilon}\right) \\
&+ \left(\sum_{i=q_n+1}^{p_{n+1}} w_i + \sum_{i=q_k+1}^{p_{k+1}} w_i\right) M\left(\frac{v}{1+\varepsilon}\right) \\
&\leq 1 + 2(1+\delta)M(c) + 2(m+\delta)\varepsilon M(v) \leq 1 + 4M(2\varepsilon) + 4\varepsilon.
\end{aligned}$$

Therefore $A(\{z^{(n)}\}_{n=1}^{\infty}) \leq (1+\varepsilon)(1+4M(2\varepsilon)+4\varepsilon)$. □

Lemma 4.4 *Let $(\ell_M(w), |\cdot|)$ be a weighted Orlicz sequence space, generated by an Orlicz function M and a weight sequence $w \in \Lambda_\infty$, such that $\ell_M(w) \not\cong h_M(w)$. Then for any $\varepsilon > 0$ there exists a sequence $\{z^{(n)}\}_{n=1}^{\infty}$, $|z^{(n)}| = 1$ such that*

$$A(\{z^{(n)}\}) \leq (1+\varepsilon)(1+\varepsilon).$$

Proof: Let the a_n be the solution of the equation $w_n M(a_n) = 1$, $n \in \mathbb{N}$. By $\ell_M(w) \not\cong h_M(w)$ it follows that for every $\varepsilon > 0$ there exists $x = \{x_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} w_i M(a_i x_i) = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} w_i M\left(\frac{a_i x_i}{1+\varepsilon}\right) < \infty. \tag{9}$$

By (9) there is a block basic sequence $y^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i e_i$, so that $1 \leq \widetilde{M_w}(y^{(n)}) < 2$ and there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ holds $\widetilde{M_w}\left(\dfrac{y^{(n)}}{1+\varepsilon}\right) < \varepsilon/2$.

Put $z^{(n)} = \dfrac{y^{(n)}}{|y^{(n)}|_M}$. By Lemma 4.2 the sequence $\{z^{(n)}\}_{n=1}^{\infty}$ is weakly null normalized sequence. Then for any $n, k \geq n_0$, $n \neq k$

$$\begin{aligned} \left|\frac{z^{(n)} - z^{(k)}}{1+\varepsilon}\right|_M &= \left|\frac{1}{1+\varepsilon}\left(\frac{y^{(n)}}{|y^{(n)}|_M} + \frac{y^{(k)}}{|y^{(k)}|_M}\right)\right|_M \leq \left|\frac{y^{(n)} + y^{(k)}}{1+\varepsilon}\right|_M \\ &\leq 1 + \sum_{i=p_n+1}^{p_{n+1}} w_i M\left(\frac{a_i x_i}{1+\varepsilon}\right) + \sum_{i=p_k+1}^{p_{k+1}} w_i M\left(\frac{a_i x_i}{1+\varepsilon}\right) \leq 1+\varepsilon. \end{aligned}$$

Therefore $A(\{z^{(n)}\}) \leq (1+\varepsilon)(1+\varepsilon)$. □

5 Proof of the main result

Sufficiency: Suppose that $\ell_M(w) \ncong h_M(w)$. If $w \in \Lambda$ or $w \in \Lambda_\infty$ then by Lemma 4.3 or Lemma 4.4, respectively it follows that $WCS((\ell_M(w), |\cdot|_M)) = 1$. Consequently by (1) it follows $WCS((\ell_M(w), \|\cdot\|_M)) \leq WCS((\ell_M(w), |\cdot|_M)) = 1$.

Necessity: Let $(\ell_M(w), \|\cdot\|_M)$ has not weak uniform normal structure i.e. $WCS((\ell_M(w), \|\cdot\|_M)) = 1$.

Let first $w \in \Lambda$. Let suppose the contrary i.e. $\ell_M(w) \cong h_M(w)$. According to Theorem 1 for any $0 < \varepsilon < 1/2$ there exists a block basic sequence $\{x^{(n)}\}_{n=1}^{\infty}$

$$x^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} x_i^{(n)} e_i \in S_{(\ell_M(w), \|\cdot\|_M)} \tag{10}$$

such that $1 \leq A(\{x^{(n)}\}) \leq 1 + \varepsilon/2$ and $x^{(n)} \xrightarrow{w} 0$. By the definition of $A(\{x^{(n)}\})$ it follows that for every $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ such that for every $m, s \geq N_1$, $m \neq s$ the inequality $\|x^{(m)} - x^{(s)}\|_M \leq A(\{x^{(m)}\}) + \varepsilon/2 \leq 1 + \varepsilon$ holds.

By $\ell_M(w) \cong h_M(w)$ it follows that M has the Δ_2–condition, therefore for some $q > \beta_M$ the inequality $M(uv) \geq u^q M(v)$ holds for every $u, v \in [0, 1]$. There exists $\delta > 0$ such that $\left(\dfrac{1}{1+\delta}\right)^q \geq 1/2$.

Hence for $m, s \geq N_1$, $m \neq s$ and by the definition of the Luxemburg's norm in $\ell_M(w)$ we get the chain of inequalities:

$$\begin{aligned} \widetilde{M}_w\left(\frac{x^{(m)} - x^{(s)}}{1+\delta}\right) &= \sum_{i=p_m+1}^{p_{m+1}} w_i M\left(\frac{x_i^{(m)}}{1+\delta}\right) + \sum_{i=p_s+1}^{p_{s+1}} w_i M\left(\frac{x_i^{(s)}}{1+\delta}\right) \\ &\geq \left(\frac{1}{1+\delta}\right)^q \left(\widetilde{M}_w(x^{(m)}) + \widetilde{M}_w(x^{(s)})\right) \geq 1. \end{aligned}$$

Thus $1 + \varepsilon \geq \|x^{(m)} - x^{(s)}\| \geq 1 + \delta$, which is a contradiction with the choice of an arbitrary small $\varepsilon > 0$. Therefore $\ell_M(w) \ncong h_M(w)$.

Let now $w \in \Lambda_\infty$. Let suppose the contrary i.e. $\ell_M(w) \cong h_M(w)$. By the fact that $\ell_M(w)$ is isometric to ℓ_φ and Theorem 1 it follows that for any $0 < \varepsilon < 1/2$ there exists a block basic sequences $\{x^{(n)}\}_{n=1}^{\infty}$ and $\{y^{(n)}\}_{n=1}^{\infty}$

$$y^{(n)} = \sum_{i=1}^{\infty} x_i e_i \in S_{\ell_\varphi} \qquad x^{(n)} = \sum_{i=1}^{\infty} a_i x_i e_i \in S_{\ell_M(w)}, \tag{11}$$

such that $1 \le A(\{x^{(n)}\}) = A(\{y^{(n)}\}) \le 1+\varepsilon/2$ and $x^{(n)}, y^{(n)} \xrightarrow{w} 0$. By the definition of $A(\{x^{(n)}\})$ it follows that for every $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ such that for every $m, s \ge N_1$, $m \ne s$ the inequality $\|x^{(m)} - x^{(s)}\|_M \le A(\{x^{(m)}\}) + \varepsilon/2 \le 1 + \varepsilon$ holds.

According to [26] $\ell_M(w) \cong \ell_\Psi$, where Ψ is defined in (6). By the fact that Ψ_i satisfies (4) for every $i \in \mathbb{N}$ it follows that for some q the inequality $\Psi_i(uv) \ge u^q \Psi_i(v)$ holds for every $u, v \in [0,1]$ and every $i \in \mathbb{N}$ ([17] p.140). There exists $\delta > 0$ such that $\left(\dfrac{1}{1+\delta}\right)^q \ge \dfrac{3}{4}$.

By the convergence of $\sum_{i=1}^\infty \varphi_i(y_i)$ it follows that there is $N_2 \in \mathbb{N}$, $N_2 \ge N_1$, such that for every $m \ge N_2$ holds $\displaystyle\sum_{i=m}^\infty \varphi_i(y_i) < 1/2$. Observe that $p_m \ge m$ and thus $\displaystyle\sum_{i=p_m}^\infty \varphi_i(y_i) \le \sum_{i=m}^\infty \varphi_i(y_i) < 1/2$ for every $m \ge N_2$.

Hence for $m, s \ge N_2$, $m \ne s$ and by the definition of the Luxemburg norm in $\ell_M(w)$ and the definition of the MO function Ψ in (6) we get the chain of inequalities

$$\begin{aligned}
\widetilde{M}_w\left(\frac{x^{(m)} - x^{(s)}}{1+\delta}\right) &= \sum_{i=p_m+1}^{p_{m+1}} w_i M\left(\frac{x_i^{(m)} a_i}{1+\delta}\right) + \sum_{i=p_s+1}^{p_{s+1}} w_i M\left(\frac{x_i^{(s)} a_i}{1+\delta}\right) \\
&= \sum_{i=p_m+1}^{p_{m+1}} \varphi_i\left(\frac{x_i^{(m)}}{1+\delta}\right) + \sum_{i=p_s+1}^{p_{s+1}} \varphi_i\left(\frac{x_i^{(s)}}{1+\delta}\right) \\
&\ge \sum_{i=p_m+1}^{p_{m+1}} \Psi_i\left(\frac{x_i^{(m)}}{1+\delta}\right) + \sum_{i=p_s+1}^{p_{s+1}} \Psi_i\left(\frac{x_i^{(s)}}{1+\delta}\right) \\
&- \sum_{i=p_m+1}^{p_{m+1}} \varphi_i(y_i) - \sum_{i=p_s+1}^{p_{s+1}} \varphi_i(y_i) \\
&\ge \left(\frac{1}{1+\delta}\right)^q \left(\sum_{i=p_m+1}^{p_{m+1}} \Psi_i(x_i^{(m)}) + \sum_{i=p_s+1}^{p_{s+1}} \Psi_i(x_i^{(s)})\right) - 1/2 \\
&\ge \left(\frac{1}{1+\delta}\right)^q \left(\sum_{i=p_m+1}^{p_{m+1}} \varphi_i(x_i^{(m)}) + \sum_{i=p_s+1}^{p_{s+1}} \varphi_i(x_i^{(s)})\right) - 1/2 \\
&= 2\left(\frac{1}{1+\delta}\right)^q - 1/2 \ge 2\frac{3}{4} - \frac{1}{2} = 1.
\end{aligned}$$

Thus $1 + \varepsilon \ge \|x^{(n)} - x^{(s)}\| \ge 1 + \delta$, which is a contradiction with the choice of an arbitrary small $\varepsilon > 0$. So $\ell_M(w) \ncong h_M(w)$.

If $(\ell_M(w), |\cdot|_M)$ has not weak uniform normal structure by the inequality $WCS((\ell_M(w), \|\cdot\|_M)) \le WCS((\ell_M(w), |\cdot|_M)) = 1$ it follows that $(\ell_M(w), \|\cdot\|_M)$ has not weak uniform normal structure and therefore $\ell_M(w) \ncong h_M(w)$. $\square$

Remark: Following [6] a direct proof in the case of Amemiya's norm can be done to show that if $(\ell_M(w), |\cdot|_M)$ has not weak uniform normal structure then $\ell_M(w) \ncong h_M(w)$.

Corollary 5.1 *Let M be an Orlicz function such that $\lim\limits_{t\to 0} \dfrac{M(t)}{t} = 0$ and $w = \{w_i\}_{i=1}^\infty$ be a weight sequence form the class Λ. Then the weighted Orlicz sequence space $\ell_M(w)$*

endowed with the Luxemburg or Amemiya norm has weak uniform normal structure iff M has Δ_2–condition.

6 Examples of weighted Orlicz sequence spaces with weak uniform normal structure without Δ_2–condition

Let N be an Orlicz function, $\lim_{t\to 0} \frac{N(t)}{t} = 0$, $N(1) = 1$ and $\{w_k\}_{k=1}^{\infty}, w_1 = 1$, be a weight sequence from the class Λ_∞ such that

$$\sum_{k=1}^{\infty} \frac{w_k}{w_{k+1}} < \infty. \tag{12}$$

Denote $a_k = N^{-1}(1/w_k)$ and choose a sequence $\{b_k\}_{k=1}^{\infty}$ fulfilling

$$0 < a_{k+1} < b_k < a_k < 1 \tag{13}$$

$$\sum_{k=1}^{\infty} \frac{N(b_k)}{N(a_k)} < \infty \tag{14}$$

$$\lim_{k\to\infty} \frac{b_k}{a_k} = 0. \tag{15}$$

We define the Orlicz function M by:

$$M(t) = \begin{cases} N(t), & t \in [a_{k+1}, b_k], \quad k \in \mathbb{N} \\ l_k(t), & t \in [b_k, a_k], \quad\ \ k \in \mathbb{N}, \end{cases} \tag{16}$$

where the line l_k is defined by $l_k(t) = \frac{N(a_k) - N(b_k)}{a_k - b_k}(t - a_k) + N(a_k)$.

For every $\alpha > 1$ there exists $n = n(\alpha)$ such that for every $k \geq n$ the inequalities $a_{k+1} \leq b_k < \frac{a_k}{\alpha} < a_k$ hold.

Let notice that by (14) it follows that

$$\lim_{k\to\infty} \frac{M(b_k)}{M(a_k)} = 0 \tag{17}$$

and by $\lim_{t\to 0} \frac{N(t)}{t} = 0$ it follows that $\lim_{t\to 0} \frac{M(t)}{t} = 0$. Indeed by $\lim_{t\to 0} \frac{M(a_k)}{a_k} = 0$ it follows that for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ holds $\frac{M(a_k)}{a_k} < \varepsilon$. Thus for every $0 < t \leq a_{k_0}$ we have $\frac{M(t)}{t} < \varepsilon$.

Throughout this paragraph by M we denote the Orlicz function defined in (16).

Lemma 6.1 *For every $0 < \lambda < 1$ we have*

$$\lim_{k\to\infty} \frac{M(\lambda a_k)}{M(a_k)} = \lambda. \tag{18}$$

Proof: For every $\lambda < 1$ there exists $i_0 \in \mathbb{N}$ such that $b_i < \lambda a_i < a_i$ holds for every $i \geq i_0$. By the chain of equalities

$$\begin{aligned}
\lim_{k\to\infty} \frac{M(\lambda a_k)}{M(a_k)} &= \lim_{k\to\infty} \frac{l_k(\lambda a_k)}{N(a_k)} \\
&= \lim_{k\to\infty} \left(\frac{N(a_k)(\lambda a_k - a_k)}{N(a_k)(a_k - b_k)} - \frac{N(b_k)(\lambda a_k - a_k)}{N(a_k)(a_k - b_k)} + \frac{N(a_k)}{N(a_k)} \right) \\
&= \lim_{k\to\infty} \left(\frac{\lambda - 1}{1 - \dfrac{b_k}{a_k}} - \frac{N(b_k)(\lambda - 1)}{N(a_k)(1 - \dfrac{b_k}{a_k})} + 1 \right) \\
&= \lim_{k\to\infty} \left(\frac{1}{1 - \dfrac{b_k}{a_k}} \left(\lambda - 1 - \frac{N(b_k)}{N(a_k)}\lambda + \frac{N(b_k)}{N(a_k)} + 1 - \frac{b_k}{a_k} \right) \right) = \lambda
\end{aligned}$$

it follows (18). $\square$

Proposition 6.1 *The weighted Orlicz sequence space $\ell_M(w)$ has weak uniform normal structure iff*

$$\sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty. \tag{19}$$

Proof: In view of the Theorem 2 we need to prove only that $\ell_M(w) \cong h_M(w)$ iff (19) holds.

Let holds (19). Let $z = \sum_{k=1}^{\infty} z_k e_k \in \ell_M(w)$ i.e. $\widetilde{M}_w(z) < \infty$. If $\widetilde{M}_w(\lambda_1 z) < \infty$ for some $\lambda_1 < 1$ we can consider the vector $\overline{z} = \lambda_1 z$.

Denote $\lambda = \max \left\{ \dfrac{b_k}{a_k} : k \in \mathbb{N} \right\}$ and let $\alpha > 1$ be arbitrary.

1) Let $I_1 = \left\{ k \in \mathbb{N} : z_k < \dfrac{b_k}{\alpha} \right\}$. Then

$$\sum_{k\in I_1} w_k M(\alpha z_k) \leq \sum_{k\in I_1} w_k M(b_k) \leq \sum_{k=1}^{\infty} w_k M(b_k) = \sum_{k=1}^{\infty} \frac{M(b_k)}{M(a_k)} < \infty.$$

2) Let $I_2 = \left\{ k \in \mathbb{N} : \dfrac{b_k}{\alpha} \leq z_k < b_k \right\}$. Then

$$\sum_{k\in I_2} w_k M(\alpha z_k) \leq \sum_{k\in I_2} w_k M(\alpha b_k) \leq \sum_{k=1}^{\infty} w_k M(\alpha b_k).$$

By (15) it follows that for every $\alpha > 1$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ holds $\alpha b_k \leq a_k$. Hence

$$\sum_{k=1}^{k_0-1} w_k M(\alpha b_k) < \infty \tag{20}$$

and

$$\sum_{k=k_0}^{\infty} w_k \quad M \quad (\alpha b_k) = \sum_{k=k_0}^{\infty} w_k \left(\frac{M(a_k)-M(b_k)}{a_k-b_k}(\alpha b_k - a_k) + M(a_k) \right)$$

$$= \sum_{k=k_0}^{\infty} w_k \frac{(\alpha-1)b_k M(a_k) - \alpha b_k M(b_k) + a_k M(b_k)}{a_k - b_k}$$

$$= \sum_{k=k_0}^{\infty} w_k \frac{(\alpha-1)\dfrac{b_k}{a_k}M(a_k) - \alpha\dfrac{b_k}{a_k}M(b_k) + M(b_k)}{1-\dfrac{b_k}{a_k}}$$

$$\leq (\alpha-1)\sum_{k=k_0}^{\infty} w_k \frac{\dfrac{b_k}{a_k}M(a_k)}{1-\dfrac{b_k}{a_k}} + \sum_{k=k_0}^{\infty} w_k M(b_k)\left(\frac{\alpha\dfrac{b_k}{a_k}+1}{1-\dfrac{b_k}{a_k}}\right) \tag{21}$$

$$\leq (1-\lambda)^{-1}\left(\sum_{k=k_0}^{\infty}\left((\alpha-1)w_k\frac{b_k}{a_k}M(a_k) + w_k M(b_k)\left(\alpha\frac{b_k}{a_k}+1\right)\right)\right)$$

$$\leq (1-\lambda)^{-1}\left((\alpha-1)\sum_{k=k_0}^{\infty}\frac{b_k}{a_k} + (\alpha\lambda+1)\sum_{k=k_0}^{\infty} w_k M(b_k)\right) < \infty.$$

By (20) and (21) it follows $\sum_{k\in I_2} w_k M(\alpha z_k) < \infty$.

3) Let $I_3 = \left\{k\in\mathbb{N} : b_k \leq z_k < \dfrac{a_k}{\alpha}\right\}$.

Obviously for $k \in I_3$

$$M(\alpha z_k) = \frac{(\alpha z_k - b_k)M(a_k) + (a_k - \alpha z_k)M(b_k)}{a_k-b_k}$$

$$= \alpha\frac{(z_k-b_k)M(a_k) + (a_k-z_k)M(b_k)}{a_k-b_k}$$

$$+ (\alpha-1)\left(\frac{b_k}{a_k-b_k}M(a_k) - \frac{a_k}{a_k-b_k}M(b_k)\right)$$

$$= \alpha M(z_k) + (\alpha-1)\left(\frac{b_k}{a_k-b_k}M(a_k) - \frac{a_k}{a_k-b_k}M(b_k)\right)$$

$$\leq \alpha M(z_k) + \frac{(\alpha-1)b_k}{(1-\lambda)a_k}M(a_k).$$

Therefore

$$\sum_{k\in I_3} w_k M(\alpha z_k) \leq \alpha\sum_{k\in I_3} w_k M(z_k) + \frac{\alpha-1}{1-\lambda}\sum_{k\in I_3}\frac{b_k}{a_k} < \infty.$$

4) Let $I_4 = \left\{k\in\mathbb{N} : \frac{a_k}{\alpha} \leq z_k\right\}$, then $\alpha z_k \geq a_k$ for every $k \in I_4$. To finish the proof we need to show that $\sum_{k\in I_4} w_k M(\alpha z_k) < \infty$.

Let us point out that in this case ($k \in I_4$) we do not know the exact definition of $M(\alpha z_k)$. By (16) there are two possibilities:
1) If $\alpha z_k \in [a_m, b_{m-1}]$ for some $m\in\mathbb{N}$, $m \leq k$, then $M(\alpha z_k) = N(\alpha z_k)$;
2) If $\alpha z_k \in [a_{m-1}, b_{m-1}]$ for some $m\in\mathbb{N}$, $m\leq k$, then $M(\alpha z_k) = l_{m-1}(\alpha z_k)$.

That's why we could not make a direct estimation of the sum $\sum_{k\in I_4} w_k M(\alpha z_k)$, as like as, it was done in the first three cases ($k \in I_1$, $k \in I_2$, $k \in I_3$). So we will prove that the sum $\sum_{k\in I_4} w_k M(\alpha z_k)$ is finite by proving that the set I_4 is finite.

Claim 6.1 *The set I_4 is finite iff the sum $\sum_{k\in I_4} w_k M(z_k)$ is finite.*

PROOF OF CLAIM 6.1: If I_4 is finite it is obvious that $\sum_{k\in I_4} w_k M(z_k)$ is finite.

For the proof of the converse let suppose the contrary i.e. $\sum_{k\in I_4} w_k M(z_k) < \infty$, but $|I_4| = \infty$. Then by Lemma 6.1 it follows that there exists $k_0 \in \mathbb{N}$ so that the inequality $\dfrac{M\left(\frac{a_k}{\alpha}\right)}{M(a_k)} \geq \dfrac{1}{2\alpha}$ holds for infinite number of indices, fulfilling $k \in I_4$ and $k \geq k_0$. Therefore

$$\sum_{k\in I_4} w_k M(z_k) \geq \sum_{k\in I_4} \frac{M\left(\frac{a_k}{\alpha}\right)}{M(a_k)} \geq \sum_{k\in I_4, k\geq k_0} \frac{M\left(\frac{a_k}{\alpha}\right)}{M(a_k)} \geq \sum_{k\in I_4, k\geq k_0} \frac{1}{2\alpha} = \infty,$$

which is a contradiction and consequently $|I_4| < \infty$. □

Thus $\sum_{k\in I_4} w_k M(\alpha z_k) < \infty$, because by Claim 6.1 the index set I_4 consists of finite number of elements.

Consequently

$$\begin{aligned}\sum_{k=1}^{\infty} w_k M(\alpha z_k) &= \sum_{k\in I_1} w_k M(\alpha z_k) + \sum_{k\in I_2} w_k M(\alpha z_k) \\ &+ \sum_{k\in I_3} w_k M(\alpha z_k) + \sum_{k\in I_4} w_k M(\alpha z_k) < \infty.\end{aligned}$$

Let now

$$\sum_{k=1}^{\infty} \frac{b_k}{a_k} = \infty. \tag{22}$$

Let $z = \{b_k\}_{k=1}^{\infty}$. By (14) follows that $z \in \ell_M(w)$. Let consider the sequence $2z = \{2b_k\}_{k=1}^{\infty}$. By (17) there exist $n \in \mathbb{N}$ such that $M(b_k)/M(a_k) < 1/4$ for every $k \geq n$. Then

$$\begin{aligned}\widetilde{M}_w(2z) &= \sum_{k=1}^{\infty} w_k M(2b_k) = \sum_{k=1}^{\infty} w_k \left(\frac{M(a_k) - M(b_k)}{a_k - b_k}(2b_k - a_k) + M(a_k)\right) \\ &= \sum_{k=1}^{\infty} w_k \left(\frac{b_k M(a_k) - 2b_k M(b_k) + a_k M(b_k)}{a_k - b_k}\right) \\ &= \sum_{k=1}^{\infty} w_k \left(\frac{M(b_k) + \frac{b_k}{a_k} M(a_k) - 2\frac{b_k}{a_k} M(b_k)}{1 - \frac{b_k}{a_k}}\right) \\ &\geq \sum_{k=n}^{\infty} \frac{w_k}{1 - \frac{b_k}{a_k}} \left(M(b_k) + \frac{1}{2}\frac{b_k}{a_k} M(a_k)\right) \\ &\geq \sum_{k=n}^{\infty} w_k M(b_k) + \frac{1}{2}\sum_{k=n}^{\infty} w_k \frac{b_k}{a_k} M(a_k) = \sum_{k=n}^{\infty} w_k M(b_k) + \frac{1}{2}\sum_{k=n}^{\infty} \frac{b_k}{a_k} = \infty.\end{aligned}$$

Therefore $\ell_M(w) \ncong h_M(w)$. $\square$

Example 1: Let $N(t) = t^2 e^{-\frac{1}{2t}}$ and $w_k = \dfrac{1}{N(1/k^{2k})}$, $k \in \mathbb{N}$. We define the sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ by $a_k = \dfrac{1}{k^{2k}}$, $b_k = \dfrac{2}{(k+1)^{2(k+1)}}$. Obviously the sequences $\{w_k\}_{k=1}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ fulfill conditions (12), (13), (14), (15) and $\sum_{k=1}^{\infty} \dfrac{b_k}{a_k} < \infty$ and by Proposition 6.1 it follows that $\ell_M(w)$ has weak uniform normal structure.

Example 2: Let $N(t) = t^2 e^{-\frac{1}{2t}}$ and $w_k = \dfrac{1}{N(1/k!)}$, $k \in \mathbb{N}$. We define the sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ by $a_k = \dfrac{1}{k!}$, $b_k = \dfrac{2}{(k+1)!}$. Obviously the sequences $\{w_k\}_{k=1}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ fulfill conditions (12), (13), (14), (15) and $\sum_{k=1}^{\infty} \dfrac{b_k}{a_k} = \infty$ and by Proposition 6.1 it follows that $\ell_M(w)$ has not weak uniform normal structure.

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