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Best Proximity Points for p–Summing Cyclic Orbital Meir–Keeler Contractions

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Abstract

We generalize the cyclic orbital Meir–Keeler contractions, which were introduced by S. Karpagam and Sushama Agrawal in the context of p–summing maps. We found sufficient conditions for these new type of maps, that ensure the existence and uniqueness of fixed points in complete metric spaces, when the distances between the sets are zero, and the existence and uniqueness of best proximity points in uniformly convex Banach spaces.

Keywords: best proximity point, Meir–Keeler contraction, cyclic orbital map, uniformly convex Banach space.

1 Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notion of cyclic maps [1]. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Interesting application of cyclic maps to integro-differential equations is presented in [2]. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of best proximity point is introduced in [3]. This definition is more general than the notion of cyclic maps [1], in sense that if the sets intersect then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [3]. We would like to mention just a few recent results in this new field [4], [5], [6].

Cyclic Meir–Keeler contractions were investigated in [7]. A cyclic orbital Meir–Keeler contraction was introduced in [8] and sufficient conditions are found for the existence of fixed points and best proximity points for these type of maps. The notion of p–summing maps was introduced in [9] and sufficient conditions are found so that these maps to have fixed points and best proximity points. The p–summing maps are wider class of maps than the classical contraction maps and cyclic contraction maps [9]. A disadvantage of the classical results about best proximity points is that the conditions are so restrictive that the distances between the successive sets are equal. The p–summing maps overcome this disadvantage [9].

S. Karpagam proposed us to try to generalize the notion of cyclic orbital Meir–Keeler contraction from [8] to the notion of p–summing cyclic contraction, that were introduced in [9]. We have succeed in obtaining of sufficient conditions for fixed points and best proximity points for such maps.

2 Preliminary results

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X, ρ) be a metric space. Define a distance between two subset $A, B \subset X$ by $\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$.

Let $\{A_i\}_{i=1}^p$ be nonempty subsets of a metric space (X, ρ). We use the convention $A_{p+i} = A_i$ for every $i \in \mathbb{N}$. The map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called a cyclic map if $T(A_i) \subseteq T(A_{i+1})$ for every $i = 1, 2, \dots p$. A point $\xi \in A_i$ is called a best proximity point of the cyclic map T in A_i if $\rho(\xi, T\xi) = \operatorname{dist}(A_i, A_{i+1})$.

Let $\{A_i\}_{i=1}^p$ be nonempty subsets of a metric space (X, ρ). The map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called p-cyclic contraction, if T is a cyclic map and for some $k \in (0, 1)$ there holds the inequality $\rho(Tx, Ty) \leq k\rho(x, y) + (1 - k)\operatorname{dist}(A_i, A_{i+1})$, for any $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq p$. The definition for 2-cyclic contraction is introduced in [3], and for p-cyclic contraction is introduced in [10]. A generalization of the cyclic maps for Meir–Keeler contractions is given in [8].

The best proximity results need norm-structure of the space X. When we investigate a Banach space $(X, \|\cdot\|)$ we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$.

The assumption that the Banach space $(X, \|\cdot\|)$ is uniformly convex plays a crucial role in the investigation of best proximity points.

Definition 2.1. *([11], p. 61) The norm $\|\cdot\|$ on a Banach space X is said to be uniformly convex if* $\lim_{n\to\infty} \|x_n - y_n\| = 0$ *whenever* $\|x_n\| = \|y_n\| = 1$, $n \in \mathbb{N}$ *are such that* $\lim_{n\to\infty} \|x_n + y_n\| = 2$.

We will use the following two lemmas for proving the uniqueness of the best proximity points.

Lemma 2.1. *([3]) Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences in A and $\{y_n\}_{n=1}^\infty$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$*;*
2) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$ there holds the inequality $\|x_m - y_n\| \leq \operatorname{dist}(A, B) + \varepsilon$.

Then for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, there holds the inequality $\|x_m - z_n\| \leq \varepsilon$.

Lemma 2.2. *([3]) Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$;
2) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$.
Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

3 Main result

Let $\{A_i\}_{i=1}^{p}$ be non empty subsets of the metric space (X, ρ). We will use the notions $P = \sum_{i=1}^{p} \operatorname{dist}(A_i, A_{i+1})$ and

$$s_p(x_1, x_2, \dots, x_p) = \sum_{j=1}^{p-1} \rho(x_j, x_{j+1}) + \rho(x_p, x_1), \tag{1}$$

where if $x_1 \in A_i$, then $x_{1+k} \in A_{i+k}$ for every $k = 1, 2, \dots, p-1$.

Definition 3.1. *Let A_i, $i = 1, 2 \dots, p$ be subsets of a metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a cyclic map. The map T is called a* **p–summing cyclic orbital Meir–Keeler contraction** *if there exists $x \in A_1$ with the property:*

$$\begin{array}{c} \textit{for every } \varepsilon > 0 \textit{ there exists } \delta > 0 \textit{ such that if there holds the inequality} \\ s_p(T^{pn-1}x, y_1, y_2, \dots, y_{p-1}) < P + \varepsilon + \delta \\ \textit{for } n \in \mathbb{N} \textit{ and } y_i \in A_i, i = 1, 2 \dots, p-1, \textit{then there holds the inequality} \\ s_p(T^{pn}x, Ty_1, Ty_2, \dots, Ty_{p-1}) < P + \varepsilon. \end{array} \tag{2}$$

If $p = 2$ in Definition 3.1 we get the definition of cyclic orbital Meir–Keeler contraction from [8].

We will introduce a new condition, which is similar to (2).

Let A_i, $i = 1, 2 \dots, p$ be subsets of a metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a cyclic map. Let there exists $x \in A_1$ with the property:

$$\begin{array}{c} \textit{for every } \varepsilon > 0 \textit{ there exists } \delta > 0 \textit{ such that if there holds the inequality} \\ s_p(T^{pn}x, y_2, y_3, \dots, y_p) < P + \varepsilon + \delta \\ \textit{for } n \in \mathbb{N} \textit{ and } y_i \in A_i, i = 2, 3 \dots, p, \textit{then there holds the inequality} \\ s_p(T^{pn+1}x, Ty_2, Ty_3, \dots, Ty_p) < P + \varepsilon. \end{array} \tag{3}$$

Theorem 3.1. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p–summing cyclic orbital Meir–Keeler contraction. Then there exists a unique point,*

say $\xi \in A_1$, such that:
a) for every $x \in A_1$, satisfying (2), the sequence $\{T^{pn}x\}$ converges to ξ;
b) ξ is a best proximity point of T in A_1;
c) $T^j\xi$ is a best proximity point of T in A_{j+1} for any $j = 1, 2, \ldots, p-1$.

If the map T satisfies (3) or T is a continuous map then ξ is a fixed point for the map T^p.

4 Auxiliary results

Definition 4.1. *Let A_i, $i = 1, 2, \ldots p$ be subsets of a metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a cyclic map. The map T is called* **p–cyclic orbital contraction** *if there exist $x \in A_1$ and $k = k(x) \in (0, 1)$ such that the inequality*

$$s_p(T^{pn}x, Ty_1, Ty_2, \ldots, Ty_{p-1}) \leq k s_p(T^{pn-1}x, y_1, y_2, \ldots, y_{p-1}) \tag{4}$$

holds for every $n \in \mathbb{N}$ and every $y_i \in A_i$, $i = 1, 2 \ldots, p-1$.

If $p = 2$ we get the definition of cyclic orbital contraction from [8].

From the definition of s_p it is easy to see that for any $x_{n_j} \in A_{i+j-1}$, $j = 1, 2, \ldots, p$ there holds the equality

$$s_p(x_{n_1}, x_{n_2}, \ldots, x_{n_p}) = s_p(x_{n_p}, x_{n_1}, x_{n_2}, \ldots, x_{n_{p-1}}). \tag{5}$$

For any $n \in \mathbb{N}$ one of the numbers $\{n + j\}_{j=0}^{p-1}$ is a multiple of p. Let $n + p - k + 1$ be a multiple of p. Applying (5) and (4) we get the inequality

$$\begin{aligned}
\alpha &= s_p(T^n x, T^{n+1}x, T^{n+2}x, \ldots, T^{n+p-1}x) \\
&= s_p(T^{n+p-k+1}x, T^{n+p-k+2}x, \ldots, T^{n+p-1}, T^n x, T^{n+1}x, \ldots, T^{n+p-k}x) \\
&\leq k s_p(T^{n+p-k}x, T^{n+p-k+1}x, \ldots, T^{n+p-2}, T^{n-1}x, T^n x, \ldots, T^{n+p-k-1}x) \\
&= k s_p(T^{n-1}x, T^n x, T^{n+1}x, \ldots, T^{n+p-2}x).
\end{aligned} \tag{6}$$

Proposition 4.1. *Let A_i, $i = 1, 2, \ldots, p$ be nonempty and closed subsets of a complete metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p–cyclic orbital contraction. Then $\bigcap_{i=1}^{p} A_i$ is nonempty and T has a unique fixed point $\xi \in \bigcap_{i=1}^{p} A_i$.*

Proof. From the condition that T is p–cyclic orbital contraction we can choose $x \in A_1$, which satisfies (4). For any $n \in \mathbb{N}$ one of the numbers $\{n + j\}_{j=0}^{p-1}$ is multiple of p, thus by applying n–times inequality (6) we can write the chain of inequalities

$$\begin{aligned}
\rho(T^n x, T^{n+1}x) &\leq s_p(T^n x, T^{n+1}x, T^{n+2}x, \ldots, T^{n+p-1}x) \\
&\leq k s_p(T^{n-1}x, T^n x, T^{n+1}x, \ldots, T^{n+p-2}x) \\
&\leq k^2 s_p(T^{n-2}x, T^{n-1}x, T^n x, \ldots, T^{n+p-3}x) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq k^n s_p(x, Tx, T^2 x, \ldots, T^{p-1}x).
\end{aligned} \tag{7}$$

Put $\alpha(x) = s_p(x, Tx, T^2x, \dots, T^{p-1}x)$. From (7) we obtain the inequality

$$\sum_{n=1}^{\infty} \rho(T^n x, T^{n+1}x) \leq \alpha(x) \sum_{n=1}^{\infty} k^n < \infty$$

and consequently the sequence $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence by the completeness of the metric space (X, ρ) it follows that there exists $\xi \in X$ such that $\lim_{n\to\infty} T^n x = \xi$. For any $j = 0, 1, \dots p-1$ the sequences $\{T^{pn+j}x\}_{n=1}^{\infty}$ are subsequences of $\{T^n x\}_{n=1}^{\infty}$ and thus $\lim_{n\to\infty} T^{pn+j}x = \xi$ for any $j = 0, 1, \dots p-1$. From the inclusions $\{T^{pn+j}x\}_{n=1}^{\infty} \subseteq A_{j+1}$ for any $j = 0, 1, \dots p-1$ and the condition that A_i, $i = 1, 2, \dots, p$ are closed sets it follows that $\xi \in \bigcap_{i=1}^{p} A_i$ and therefore $\bigcap_{i=1}^{p} A_i$ is not an empty set.

We will prove that ξ is a unique fixed point for the map T.

Put $S_1 = s_{p-1}(\xi, T\xi, T^2\xi, \dots, T^{p-2}\xi)$. From the continuity of the function $\rho(\cdot, z)$ and condition (4) we can write the inequalities

$$\begin{array}{rcl} S_1 & = & s_{p-1}(\xi, T\xi, T^2\xi, \dots, T^{p-2}\xi) \leq s_p(\xi, T\xi, T^2\xi, \dots, T^{p-2}\xi, T^{p-1}\xi) \\ & = & \lim_{n\to\infty} s_p(T^{pn}x, T\xi, T^2\xi, \dots, T^{p-2}\xi, T^{p-1}\xi) \\ & \leq & k \lim_{n\to\infty} s_p(T^{pn-1}x, \xi, T\xi, \dots, T^{p-3}\xi, T^{p-2}\xi) \\ & = & k s_p(\xi, \xi, T\xi, \dots, T^{p-3}\xi, T^{p-2}\xi) = k s_{p-1}(\xi, T\xi, T^2\xi, \dots, T^{p-2}\xi) = kS_1. \end{array}$$

Hence we obtain that $(1-k)s_{p-1}(\xi, T\xi, T^2\xi, \dots, T^{p-2}\xi) \leq 0$ and thus $\rho(\xi, T\xi) = 0$. Consequently ξ is a fixed point for the map T.

To finish the proof it remains to show that the point $\xi \in \bigcap_{i=1}^{p} A_i$ is a unique fixed point for the map T.

Suppose that there exists $\eta \neq \xi$ such that $T\eta = \eta$. By using the continuity of the function $\rho(\cdot, z)$, condition (4) and the assumption that $\rho(T^n\eta, T^m\eta) = 0$ for every $m, n \in \mathbb{N} \cup \{0\}$ we can write the inequalities

$$\begin{array}{rcl} 2\rho(\xi, \eta) & = & \rho(\xi, T\eta) + \rho(T^{p-1}\eta, \xi) = s_p(\xi, T\eta, T^2\eta, \dots, T^{p-1}\eta) \\ & = & \lim_{n\to\infty} s_p(T^{pn}x, T\eta, T^2\eta, \dots, T^{p-1}\eta) \\ & \leq & k \lim_{n\to\infty} s_p(T^{pn-1}x, \eta, T\eta, \dots, T^{p-2}\eta) \\ & = & k \lim_{n\to\infty} (\rho(T^{pn-1}x, \eta) + \rho(\eta, T^{pn-1}x)) = 2k\rho(\xi, \eta). \end{array}$$

Hence we obtain $(1-k)(\rho(\xi, \eta)) \leq 0$ and consequently it follows that $\xi = \eta$. □

Proposition 4.2. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a complete metric space (X, ρ) and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p–cyclic orbital contraction. Then T is p–summing cyclic orbital Meir–Keeler contraction.*

Proof. It follows from Proposition 4.1 that $P = 0$, because $\bigcap_{i=1}^{p} A_i \neq \emptyset$. There are $x \in A_1$ and $k = k(x) \in (0, 1)$, such that inequality (4) holds. Let $\varepsilon > 0$ be arbitrary chosen. Put $\delta = \frac{\varepsilon(1-k)}{k}$. For any $y_i \in A_i$, $i = 1, 2, \dots, p-1$ that satisfy the inequality $s_p(T^{pn-1}x, y_1, y_2, \dots, y_{p-1}) < \varepsilon + \delta$ there holds the inequality

$$s_p(T^{pn}x, Ty_1, Ty_2, \dots, Ty_{p-1}) \leq k s_p(T^{pn-1}x, y_1, y_2, \dots, y_{p-1}) < k(\varepsilon + \delta) = \varepsilon.$$

□

Proposition 4.3. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) and T be a p–summing cyclic orbital Meir–Keeler contraction. Then for any $x_1 \in A_1$, $x_2, x_3, \dots, x_p \in \bigcup_{i=1}^{p} A_i$, $n_1, n_2, \dots, n_p \in \mathbb{N}$, such that $T^{n_i}x_i \in A_i$ for $i = 1, 2, \dots, p$ and x_1 satisfies (2) there holds the inequality*

$$s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) \leq s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p). \tag{8}$$

Proof. For any $x_1, x_2, \dots, x_p \in \cup_{i=1}^{p} A_i$, $n_1, n_2, \dots, n_p \in \mathbb{N}$, that satisfy the conditions of the proposition there holds the inequality $s_p(T^{n_1-1}x_1, \dots, T^{n_p-1}x_p) \geq P$.

Case I) $s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p) = P$.

By (2) we have that for any $\varepsilon > 0$ there holds the inequality

$$s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) < P + \varepsilon.$$

By the arbitrary choice of $\varepsilon > 0$ it follows that $s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) = P$ and thus $s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) = s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p)$.

Case II) $s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p) > P$.

Put $\varepsilon_0 = s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p) - P > 0$. By (2) there exists $\delta = \delta(\varepsilon_0) > 0$, such that the inequality $s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) < P + \varepsilon_0$ holds for any $x_2, x_3, \dots, x_p$, that satisfy the conditions of the proposition and the inequality

$$s_p(T^{n_1-1}x_1, \dots, T^{n_p-1}x_p) < P + \varepsilon_0 + \delta.$$

From $\varepsilon_0 = s_p(T^{n_1-1}x_1, \dots, T^{n_p-1}x_p) - P < \varepsilon_0 + \delta$ we get that

$$s_p(T^{n_1}x_1, T^{n_2}x_2, \dots, T^{n_p}x_p) < P + \varepsilon_0 = s_p(T^{n_1-1}x_1, T^{n_2-1}x_2, \dots, T^{n_p-1}x_p).$$

From Case I) and Case II) we get that (8) is true. □

Just for the conveniens of the application of Proposition 4.3 we will state the next Corollary.

Corollary 4.1. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) and T be a p–summing cyclic orbital Meir–Keeler contraction. Let $x \in A_1$ satisfies (2). Let $x_i \in A_i$, $i = 1, 2, \dots, p-1$, $n \in \mathbb{N}$. Then there hold the inequalities*

$$s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) \leq s_p(T^{n-1}x, T^n x, \dots, T^{n+p-2}x); \tag{9}$$

$$s_p(T^{pn}x, Tx_1, \dots, Tx_{p-1}) \leq s_p(T^{pn-1}x, x_1, \dots, x_{p-1}). \tag{10}$$

Lemma 4.1. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) and T be a p–summing cyclic orbital Meir–Keeler contraction. For any $x \in A_1$ that satisfies (2) there holds $\lim_{n\to\infty} s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) = P$.*

Proof. Put $r_n = s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x)$, then $r_n \geq P$. It follows from (9) that the sequence $\{r_n\}_{n=1}^{\infty}$ is a nonincreasing sequence. Hence $\lim_{n\to\infty} r_n = r \geq P$.

We claim that $r = P$. Let us suppose the contrary, i.e. $r > P$. Put $\varepsilon_0 = r - P > 0$. There exists $\delta > 0$ such that the inequality

$$r_n = s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) < P + \varepsilon_0$$

holds whenever

$$r_{n-1} = s_p(T^{n-1}x, T^n x, \dots, T^{n+p-2}x) < P + \varepsilon_0 + \delta. \tag{11}$$

By $\lim_{n\to\infty} s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) = r$ it follow that there is $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ there holds the inequalities

$$r \leq s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) < r + \delta = \varepsilon_0 + P + \delta.$$

Therefore (11) holds for $n-1 \geq n_0$. Thus by the assumption that T is a p–summing cyclic orbital Meir–Keeler contraction the inequality

$$r_n = s_p(T^n x, T^{n+1}x, \dots, T^{n+p-1}x) < P + \varepsilon_0 = r$$

holds true for every $n \geq n_0$, which is a contradiction. Consequently $r = P$. $\square$

Remark 4.1 If $x, x_1, x_2, \dots, x_{p-1} \in A_1$ it can be proved in a similar fashion

$$\lim_{n\to\infty} s_p(T^n x, T^{n+1}x_1, T^{n+2}x_2, \dots, T^{n+p-1}x_{p-1}) = P.$$

Corollary 4.2. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) and T be a p–summing cyclic orbital Meir–Keeler contraction. Then for any $x \in A_1$ that satisfies (2) there hold*

$$\lim_{n\to\infty} \rho(T^{pn+j}x, T^{pn+j+1}x) = \operatorname{dist}(A_{j+1}, A_{j+2})$$

$$\lim_{n\to\infty} \rho(T^{pn+p+j}x, T^{pn+j+1}x) = \operatorname{dist}(A_{j+1}, A_{j+2})$$

for any $j = 0, 1, 2, \dots, p-1$, where we use the convention $A_{p+1} = A_1$.

Lemma 4.2. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) with $P = 0$. Let T be a p–summing cyclic orbital Meir–Keeler contraction. Then for any $x \in A_1$ that satisfies (2) and for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that there holds the inequality*

$$s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < \varepsilon \tag{12}$$

for any $m \geq n \geq N_0$.

Proof. We will prove Lemma 4.2 by induction on m.

Let $\varepsilon > 0$ be arbitrary. There exists $\delta > 0$, such that condition (2) holds true.

By Lemma 4.1 there exists $N_1 \in \mathbb{N}$ such that there holds the inequality

$$s_p(T^{pn}x, \dots, T^{pn+j}x, \dots, T^{pn+p-1}x) < \varepsilon$$

for every $n \geq N_1$. From Corollary 4.2 there exists $N_2 \in \mathbb{N}$, such that for every $n \geq N_2$ there hold the inequalities $\rho(T^{pn+j-2}x, T^{pn+j-1}x) < \frac{\delta}{2p}$ for $j = 1, 2, \dots, p$. Put $N_0 = \max\{N_1, N_2\}$.

Inequality (12) is true for $m = n \geq N_0$.

Let (12) holds true for some $m \geq n$.
We will prove that (12) holds true for $m+1$.
Put $S_2 = s_p(T^{pn-1}x, T^{p(m+1)}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p-2}x)$.
By Corollary 4.1 and the inductive assumption we obtain the inequalities

$$\begin{array}{rcl} S_2 & = & s_p(T^{pn-1}x, T^{p(m+1)}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p+1}x) \\ & \leq & s_p(T^{p(n+1)-1}x, T^{p(m+1)}x, \dots, T^{p(m+1)+p-2}) \\ & & +2\rho(T^{pn-1}x, T^{p(n+1)-1}x) \\ & \leq & s_p(T^{p(n+1)-1}x, T^{p(m+1)}x, \dots, T^{p(m+1)+p-2}) \\ & & +2\sum_{j=1}^{p}\rho(T^{pn+j-2}x, T^{pn+j-1}x) \\ & \leq & s_p(T^{pn}x, T^{pm+1}x, \dots, T^{pm+p-1}) \\ & & +2\sum_{j=1}^{p}\rho(T^{pn+j-2}x, T^{pn+j-1}x) \\ & < & \varepsilon + 2p\frac{\delta}{2p} = \varepsilon + \delta. \end{array} \tag{13}$$

The map T is a p–summing cyclic orbital Meir–Keeler contraction with $P = 0$ and from the choice of $x \in A_1$, $\delta > 0$ and (13) it follows that

$$s_p(T^{pn}x, T^{p(m+1)+1}x, \dots, T^{p(m+1)+p-1}x) < \varepsilon.$$

□

Corollary 4.3. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a metric space (X, ρ) with $P = 0$. Let T be a p–summing cyclic orbital Meir–Keeler contraction and $x \in A_1$ satisfies (2). Then for any $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for any $m \geq n \geq N_1$ there hold the inequalities*

$$\rho(T^{pn}x, T^{pm+1}x) < \varepsilon \text{ and } \rho(T^{pm+p-1}x, T^{pn}x) < \varepsilon.$$

Theorem 4.1. *Let A_i, $i = 1, 2, \dots, p$ be nonempty closed subsets of a complete metric space (X, ρ) such that $P = 0$. Let T be a p–summing cyclic orbital Meir–Keeler contraction. Then there exists a unique $\xi \in \bigcap_{i=1}^{p} A_i$, such that:*
a) $T\xi = \xi$;
b) for any $x \in A_1$, that satisfies (2) there holds $\lim_{n\to\infty} T^{pn}x = \xi$.

Proof. Let $x \in A_1$ satisfies (2). We claim that for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that the inequality $\rho(T^{pm}x, T^{pn}x) < \varepsilon$ holds for any $m \geq n \geq N_0$.

For any $\varepsilon > 0$ by Corollary 4.1 and Corollary 4.3 there is $N_0 \in \mathbb{N}$ such that there holds the inequality

$$\max\{\rho(T^{pn}x, T^{pm+1}x), \rho(T^{pm+1}x, T^{pm}x)\} < \varepsilon/2$$

for every $m \geq n \geq N_0$. Thus by the inequalities

$$\rho(T^{pn}x, T^{pm}x) \leq \rho(T^{pn}x, T^{pm+1}x) + \rho(T^{pm+1}x, T^{pm}x) < \varepsilon$$

it follows that the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequences and therefore by the completeness of the space (X, ρ) it follows that there exists $\xi \in X$ such that $\lim_{n\to\infty} T^{pn}x = \xi$.

By the inequality $\rho(T^{pn+1}x,\xi)\le \rho(T^{pn+1}x,T^{pn}x)+\rho(T^{pn}x,\xi)$ and Corollary 4.2 it follows that

$$\lim_{n\to\infty} T^{pn+1}x=\xi. \tag{14}$$

From the inequality $\rho(T^{pn+2}x,\xi)\le \rho(T^{pn+2}x,T^{pn+1}x)+\rho(T^{pn+1}x,\xi)$, (14) and Corollary 4.2 it follows that

$$\lim_{n\to\infty} T^{pn+2}x=\lim_{n\to\infty} T^{pn}x=\lim_{n\to\infty} T^{pn+1}x=\xi. \tag{15}$$

We can obtain in a similar fashion that $\lim_{n\to\infty} T^{pn+j}x=\lim_{n\to\infty} T^{pn}x=\xi$ holds for every $j=0,1,2,\dots,p-1$. Since A_i, $i=1,2,\dots p$ are closed sets we abtain that $\xi\in A_i$ for every $i=1,2,\dots,p$. Consequently we get that $\xi\in\bigcap_{i=1}^p A_i$.

We will prove that $T\xi=\xi$. We apply Corollary 4.1, the continuity if the function $\rho(\cdot,y)$ and (15) in the next chain of inequalities

$$\begin{array}{rcl}
\rho(\xi,T\xi) & \le & s_p(\xi,T\xi,T^2\xi,\dots,T^{p-1}\xi)\\
& = & \lim_{n\to\infty} s_p(T^{pn}x,T\xi,T^2\xi,\dots,T^{p-1}\xi)\\
& \le & \lim_{n\to\infty} s_p(T^{pn-1}x,\xi,T\xi,\dots,T^{p-2}\xi)\\
& = & \lim_{n\to\infty} s_p(T^{pn-1}x,T^{pn}x,T\xi,\dots,T^{p-2}\xi)\\
& \le & \lim_{n\to\infty} s_p(T^{pn-2}x,T^{pn-1}x,\xi,T\xi,\dots,T^{p-3}\xi).
\end{array}$$

By applying the above procedure p–times and Lemma 4.1 we get

$$\begin{array}{rcl}
\rho(\xi,T\xi) & \le & s_p(\xi,T\xi,T^2\xi,\dots,T^{p-1}\xi)\\
& \le & \lim_{n\to\infty} s_p(T^{p(n-1)}x,T^{p(n-1)+1}x,\dots,T^{p(n-1)+(p-1)}x)=0.
\end{array}$$

Thus ξ is a fixed point for the map T.

It remains to prove that ξ is unique.

Suppose that there exists $z\in A_1$, $z\neq x$, which satisfies (2). Then by what we have just proved it follows that $\{T^{pn}z\}_{n=1}^\infty$ converges to some point $\eta\in\bigcap_{i=1}^p A_i$, such that $T\eta=\eta$. By Remark 4.1, since $P=0$ it follows that

$$\lim_{n\to\infty} s_p(T^{pn}z,T^{pn+1}x,T^{pn+2}x,\dots,T^{pn+p-1}x)=0. \tag{16}$$

From the continuity of the function $\rho(\cdot,\cdot)$ and (16) we get

$$\begin{array}{rcl}
\rho(\eta,\xi) & = & \lim_{n\to\infty}\rho(T^{pn}z,T^{pn+1}x)\\
& \le & \lim_{n\to\infty} s_p(T^{pn}z,T^{pn+1}x,T^{pn+2}x,\dots,T^{pn+p-1}x)=0.
\end{array}$$

Hence $\xi=\eta$. $\square$

Lemma 4.3. *Let A_i, $i=1,2,\dots,p$ be nonempty closed subsets of a uniformly convex Banach space $(X,\|\cdot\|)$. Let T be a p–summing cyclic orbital Meir–Keeler contraction. Then for every $x\in A_1$, satisfying (2), the following statement holds*

$$\lim_{n\to\infty}\|T^{pn+j}x-T^{p(n+1)+j}x\|=0$$

for every $j=0,1,\dots,p-1$.

Proof. By Corollary 4.2 for any $j = 0, 1, \dots, p-1$ it follows that

$$\lim_{n\to\infty} \|T^{pn+j}x - T^{pn+j+1}x\| = \operatorname{dist}(A_{j+1}, A_{j+2})$$

and

$$\lim_{n\to\infty} \|T^{pn+p+j}x - T^{pn+j+1}x\| = \operatorname{dist}(A_{j+1}, A_{j+2}).$$

According to Lemma 2.2 it follows that $\lim_{n\to\infty} \|T^{pn+j}x - T^{p(n+1)+j}x\| = 0$. $\square$

Lemma 4.4. *Let A_i, $i = 1, 2, \dots p$ be nonempty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Let T be a p–summing cyclic orbital Meir–Keeler contraction. Then for any $x \in A_1$ that satisfies (2) and for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that there holds the inequality*

$$s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < P + \varepsilon \tag{17}$$

for any $m \ge n \ge \mathbb{N}_0$

Proof. We will prove by induction on m.

Let $\varepsilon > 0$ be arbitrary. There exists $\delta > 0$, such that condition (2) holds true.

By Lemma 4.1 there exists $N_1 \in \mathbb{N}$ such that there holds the inequality

$$s_p(T^{pn}x, \dots, T^{pn+j}x, \dots, T^{pn+p-1}x) < P + \varepsilon$$

for every $n \ge N_1$. By Lemma 4.3 there exists $N_2 \in \mathbb{N}$ such that there hold the inequalities $\|T^{pn-p}x - T^{pn}x\| < \delta/2$ for every $n \ge N_2$. Put $N_0 = \max\{N_1, N_2\}$.

Inequality (17) is true for $m = n \ge N_0$.

Let (17) holds true for some $m \ge n$.

We will prove that (17) holds true for $m + 1$.

Let us put $S_3 = s_p(T^{pn-p}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x)$. It is easy to observe that

$$\begin{aligned}
S_3 &= \|T^{pn-p}x - T^{pm+1}x\| + \sum_{j=pm+1}^{pm+p-2} \|T^j x, T^{j+1}x\| + \|T^{pm+p-1}x, T^{pn-p}x\| \\
&\le \|T^{pn-p}x - T^{pn}x\| + \|T^{pn}x - T^{pm+1}x\| + \sum_{j=pm+1}^{pm+p-2} \|T^j x, T^{j+1}x\| \\
&\quad + \|T^{pm+p-1}x, T^{pn}x\| + \|T^{pn-p}x - T^{pn}x\| \\
&= s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) + 2\|T^{pn-p}x - T^{pn}x\|.
\end{aligned}$$

Consequently for any $n \ge N_0$ there holds the inequality $S_3 \le P + \varepsilon + \delta$. From (5) we get the inequality

$$s_p(T^{pm+p-1}x, T^{pn-p}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-2}x) = S_3 \le P + \varepsilon + \delta.$$

Therefore from (2) it follows that

$$s_p(T^{pm+p}x, T^{pn-p+1}x, T^{p(m+1)-p+2}x, T^{p(m+1)-p+3}x, \dots, T^{p(m+1)-1}x) < P + \varepsilon.$$

Using again (5) we get

$$s_p(T^{pn-p+1}x, T^{p(m+1)-p+2}x, T^{p(m+1)-p+3}x, \dots, T^{p(m+1)}x) < P + \varepsilon.$$

Put $S_4 = s_p(T^{pn}x, T^{p(m+1)+1}x, T^{p(m+1)+2}x, \dots, T^{p(m+1)+p-1}x)$ and $S_5 = s_p(T^{pn-p+1}x, T^{p(m+1)-p+2}x, T^{p(m+1)-p+3}x, \dots, T^{p(m+1)}x)$.

From Corollary 4.1 we get the inequalities $S_4 \le S_5 < P + \varepsilon$. $\square$

Let us recall the definition of strictly convex Banach space.

Definition 4.2. *([11], p. 42) We say that the Banach space $(X, \|\cdot\|)$ is strictly convex if $x = y$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.*

Proposition 4.4. *([11], p. 42) The following conditions on a norm $\|\cdot\|$ of a Banach space X are equivalent.*
(i) The norm $\|\cdot\|$ is strictly convex.
(ii) If $x, y \in X$ are such that $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$, then $x = y$.
(iii) If $x, y \in X$ are such that $\|x + y\| = \|x\| + \|y\|$, $x \ne 0$ and $y \ne 0$, then $x = \lambda y$ for some $\lambda > 0$.

Lemma 4.5. *Let A, B be closed subsets of a strictly convex Banach space $(X, \|\cdot\|)$, such that $\operatorname{dist}(A, B) > 0$ and let A be convex. If $x, z \in A$ and $y \in B$ be such that $\|x - y\| = \|z - y\| = \operatorname{dist}(A, B)$, then $x = z$.*

Proof. There is no $\lambda > 0$, such that $z - y = \lambda(y - x)$. Indeed if there exists $\lambda > 0$, such that $z - y = \lambda(y - x)$, then $y = \frac{1}{1+\lambda}z + \frac{\lambda}{1+\lambda}x$ and consequently it follows that $y \in A$, because A is convex, which is a contradiction with the assumption that $\operatorname{dist}(A, B) > 0$. Thus according to Proposition 4.4 it follows that

$$\left\|\frac{x+z}{2} - y\right\| = \left\|\frac{x}{2} - \frac{y}{2} + \frac{z}{2} - \frac{y}{2}\right\| < \frac{1}{2}\left(\|x - y\| + \|z - y\|\right) = \operatorname{dist}(A, B).$$

Therefore there exists and element $u = \frac{x+z}{2} \in A$, such that $\|u - y\| < \operatorname{dist}(A, B)$, which is a contradiction. $\square$

Let us mention the well known fact, that any uniformly convex Banach space is strictly convex ([11], p.61).

5 Proof of main result

Let $x \in A_1$ satisfies (2).

Case I) Let $P = 0$. From Theorem 4.1 there exists a unique fixed point of T, which is a best proximity point.

Case II) Let $P > 0$. We will prove that the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequence. By Corollary 4.2 we have that $\lim_{m\to\infty} \|T^{pm}x - T^{pm+1}x\| = \operatorname{dist}(A_1, A_2)$. From Lemma 4.4 we have that for any $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$, such that there holds the inequality $s_p(T^{pn}x, T^{pm+1}x, T^{pm+2}x, \dots, T^{pm+p-1}x) < P + \varepsilon/2$ for every $m \ge n \ge N_1$. Therefore the inequality $\|T^{pn}x - T^{pm+1}x\| \le \operatorname{dist}(A_1, A_2) + \varepsilon/2$ holds for every $m \ge n \ge N_1$. According to Lemma 2.1 it follows that for any $\varepsilon > 0$

there exists $N_2 \in \mathbb{N}$, such that for any $m \geq n \geq N_2$ there holds the inequality $\|T^{pn}x - T^{pm}x\| \leq \varepsilon/2 < \varepsilon$ and thus $\{T^{pn}x\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence the sequence $\{T^{pn}x\}_{n=1}^{\infty}$ is convergent to some $\xi \in A_1$.

By Corollary 4.1, Lemma 4.1, and the continuity of the function $\|\cdot\|$ we can write the chain of inequalities

$$\begin{array}{lll} P & \leq & s_p(\xi, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\ & = & \lim_{n\to\infty} s_p(T^{pn}x, T\xi, T^2\xi, \dots, T^{p-1}\xi) \\ & \leq & \lim_{n\to\infty} s_p(T^{pn-1}x, \xi, T\xi, \dots, T^{p-2}\xi) \\ & = & \lim_{n\to\infty} s_p(T^{pn-1}x, T^{pn}x, T\xi, \dots, T^{p-2}\xi) \\ & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \leq & \lim_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}x, T^{pn-p+2}x, \dots, T^{pn-1}x) = P. \end{array} \tag{18}$$

Form (18) we get that

$$\begin{array}{c} \|\xi - T\xi\| = \operatorname{dist}(A_1, A_2),\ \|\xi - T^{p-1}\xi\| = \operatorname{dist}(A_1, A_p), \\ \|T^j\xi - T^{j+1}\xi\| = \operatorname{dist}(A_{j+1}, A_{j+2}),\ j = 1, 2, \dots p-2. \end{array}$$

Thus ξ is a best proximity point of T in A_1, $T^j\xi$, $j = 1, 2, \dots p-1$ is a best proximity point of T in A_{j+1}.

We will show that for any $z \in A_1$, $z \neq x$, such that z satisfies (2) there holds $\lim_{n\to\infty} T^{pn}z = \xi$. By what we have just proved $\{T^{pn}z\}$ converges to a best proximity point, say $\eta \in A_1$, of T in A_1. By Remark 4.1 we have

$$\lim_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}z, T^{pn-p+2}z, \dots, T^{pn-1}z) = P. \tag{19}$$

By Corollary 4.1, (19) and the continuity of the function $\|\cdot\|$ we can write the chain of inequalities

$$\begin{array}{lll} P & \leq & s_p(\xi, T\eta, T^2\eta, \dots, T^{p-1}\eta) = \lim_{n\to\infty} s_p(T^{pn}x, T\eta, T^2\eta, \dots, T^{p-1}\eta) \\ & \leq & \lim_{n\to\infty} s_p(T^{pn-1}x, \eta, T\eta, \dots, T^{p-2}\eta) \\ & = & \lim_{n\to\infty} s_p(T^{pn-1}x, T^{pn}z, T\eta, \dots, T^{p-2}\eta) \\ & \leq & \lim_{n\to\infty} s_p(T^{pn-2}x, T^{pn-1}z, \eta, T\eta, \dots, T^{p-3}\eta) \\ & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \leq & \lim_{n\to\infty} s_p(T^{pn-p}x, T^{pn-p+1}z, T^{pn-p+2}z, \dots, T^{pn-1}z) = P. \end{array}$$

Therefore we get that $\|\xi - T\eta\| = \|\xi - T\xi\| = \operatorname{dist}(A_1, A_2)$. Since A_2 is convex set in a uniformly convex Banach space it follows from Lemma 4.5 that $T\eta = T\xi$. By the fact that η is a best proximity point of T in A_1 there hold the equalities

$$\|\eta - T\eta\| = \|\eta - T\xi\| = \operatorname{dist}(A_1, A_2) = \|\xi - T\xi\|.$$

Since A_1 a convex set in a uniformly convex Banach space and $T\eta = T\xi$ it follows from Lemma 4.5 that $\eta = \xi$.

It remains to prove that $\xi = T^p\xi$.

Let T satisfies (3). From the inequality $\|T^{pn+1}x - \xi\| \leq \|T^{pn+1}x - T^{pn}x\| + \|T^{pn}x - \xi\|$ and Corollary 4.2 it follows that $\lim_{n\to\infty} \|T^{pn+1}x - \xi\| = \operatorname{dist}(A_1, A_2)$.

By Lemma 2.2 and $\|T\xi - \xi\| = \operatorname{dist}(A_1, A_2)$ we get $\lim_{n\to\infty} T^{pn+1}x = T\xi$. Let $\varepsilon > 0$ be arbitrary chosen. By $\lim_{n\to\infty} T^{pn}x = \xi$ it follows that for any $\delta > 0$ there is $N_2 \in \mathbb{N}$, such that for every $n \geq N_2$ there holds $s_p(T^{pn}x, T\xi, T^2\xi, \ldots, T^{p-1}\xi) < P + \varepsilon + \delta$. By (3) it follows that $s_p(T^{pn+1}x, T^2\xi, T^3\xi, \ldots, T^p\xi) < P + \varepsilon$. Hence $\|T^{pn+1}x - T^p\xi\| < \operatorname{dist}(A_1, A_2) + \varepsilon$ for every $n \geq N_2$. By the arbitrary choice of $\varepsilon > 0$ it follows that $\lim_{n\to\infty} \|T^{pn+1}x - T^p\xi\| = \operatorname{dist}(A_1, A_2)$. From $\lim_{n\to\infty} \|T^{pn+1}x - \xi\| = \operatorname{dist}(A_1, A_2)$ and Lemma 2.2 e get that $\|T^p\xi - \xi\| = 0$. Thus ξ is a fixed point for the map T^p.

Let T be a continuous map. By Corollary 4.2 it follows that $\lim_{n\to\infty} T^{pn-1}x = T^{p-1}\xi$. From the continuity of T we get the equalities:

$$\xi = \lim_{n\to\infty} T^{pn}x = \lim_{n\to\infty} T(T^{pn-1}x) = T(T^{p-1}\xi) = T^p\xi.$$

Hence ξ is a fixed point for the map T^p.

□

6 Examples

The main results in [8] are consequences from the above results.

Theorem 6.1. *([8], Theorem 2.2) Let A and B be nonempty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then $A \cap B$ is nonempty and T has a unique fixed point*

Proof. The proof follows from Proposition 4.1. □

Theorem 6.2. *([8], Theorem 2.11) Let X be a complete metric space and A and B be nonempty closed subsets of X, such that $\operatorname{dist}(A, B) = 0$. Let $T : A \cup B \to A \cup B$ be a cyclic orbital Meir–Keeler contraction. Then there exists a fixed point, say $\xi \in A \cap B$, such that for each $x \in A$, satisfying (2), the sequence $\{T^{2n}x\}$ converges to ξ.*

Proof. The proof follows from Theorem 4.1. □

Theorem 6.3. *([8], Theorem 2.13) Let X be a uniformly convex Banach space and A and B be nonempty closed and convex subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic orbital Meir–Keeler contraction. Then there exists a best proximity point, say $\xi \in A$, such that for every $x \in A$, satisfying (2), the sequence $\{T^{2n}x\}$ converges to ξ.*

Proof. The proof follows from Theorem 3.1. □

We would like to illustrate Theorem 3.1 by one example, which is in some sense very close to the examples in [12].

Let consider the space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$. Let $A_i \subset \mathbb{R}^2$ be defined by $A_1 = \{(x, 0) \in \mathbb{R}^2 : x \in [1, 2]\}$, $A_2 = \{(0, y) \in \mathbb{R}^2 : y \in [1, 2]\}$, $A_3 = \{(z, 0) \in \mathbb{R}^2 : z \in [-2, -1]\}$. It is easy to observe that

$$P = \operatorname{dist}(A_1, A_2) + \operatorname{dist}(A_2, A_3) + \operatorname{dist}(A_3, A_1) = 2\sqrt{2} + 2.$$

Put $x_i = (x_i^{(1)}, x_i^{(2)}) \in A_i$, $i = 1, 2, 3$. Let T be a cyclic map, $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, 3$, and $A_4 \equiv A_1$, defined by

$$T(x_1) = \begin{cases} (0,1), & x_1^{(1)} \in \mathbb{Q},\ x_1^{(1)} \neq 2 \\ \left(0, 1 + \frac{x_1^{(1)}}{8}\right), & x_1^{(1)} \notin \mathbb{Q} \\ (0,2), & x_1^{(1)} = 2; \end{cases}$$

$$T(x_2) = \begin{cases} (-1,0), & x_2^{(2)} \in \mathbb{Q},\ x_2^{(2)} \neq 2 \\ \left(-1 - \frac{x_2^{(2)}}{8}, 0\right), & x_2^{(2)} \notin \mathbb{Q} \\ (-2,0), & x_2^{(2)} = 2; \end{cases}$$

$$T(x_3) = \begin{cases} (1,0), & x_3^{(1)} \in \mathbb{Q},\ x_3^{(1)} \neq 2 \\ \left(1 + \frac{x_3^{(1)}}{8}, 0\right), & x_3^{(1)} \notin \mathbb{Q} \\ (2,0), & x_3^{(1)} = -2. \end{cases}$$

We will use the inequalities $1 + \frac{t}{4} \leq \sqrt{1+t} \leq 1 + \frac{t}{2}$, which hold for every $t \in [0,1]$.

We will show that the map T with $x \in A_1$, $x \in \mathbb{Q} \backslash \{2\}$ is a 3–summing cyclic orbital Meir–Keeler contraction. It is easy to observe that $T^{3n}x = (1,0)$, $T^{3n-1}x = (-1,0)$. Put $y_1 = (1+\alpha, 0) \in A_1$, $y_2 = (0, 1+\beta) \in A_2$, $S_{3n-1} = \|T^{3n-1}x - y_1\| + \|y_1 - y_2\| + \|y_2 - T^{3n-1}x\|$ and $S_{3n} = \|T^{3n}x - Ty_1\| + \|Ty_1 - Ty_2\| + \|Ty_2 - T^{3n}x\|$.

Let $\varepsilon > 0$ be arbitrary chosen. Put $\delta = \frac{\varepsilon}{5}$. Let y_1 and y_2 be chosen so that $S_{3n-1} < P + \varepsilon + \delta$. Then by the inequality

$$\begin{aligned} P + \tfrac{6\varepsilon}{5} &= P + \varepsilon + \delta > S_{3n-1} \\ &= 2 + \alpha + \sqrt{2\left(1 + \alpha + \beta + \tfrac{\alpha^2+\beta^2}{2}\right)} + \sqrt{2\left(1 + \beta + \tfrac{\beta^2}{2}\right)} \\ &\geq P + \alpha + \tfrac{\sqrt{2}}{4}(\alpha+\beta) + \tfrac{\sqrt{2}}{8}(\alpha^2+\beta^2) + \tfrac{\sqrt{2}}{4}\beta + \tfrac{\sqrt{2}}{8}\beta^2 \end{aligned}$$

we get the inequality $\frac{6\varepsilon}{5} > \alpha + \frac{\sqrt{2}}{4}(\alpha+\beta) + \frac{\sqrt{2}}{8}(\alpha^2+\beta^2) + \frac{\sqrt{2}}{4}\beta + \frac{\sqrt{2}}{8}\beta^2$. Therefore we can write the chain of inequalities

$$\begin{aligned} \varepsilon &> \tfrac{5}{6}\alpha + \tfrac{5}{6}\tfrac{\sqrt{2}}{4}(\alpha+\beta) + \tfrac{5}{6}\tfrac{\sqrt{2}}{8}(\alpha^2+\beta^2) + \tfrac{5}{6}\tfrac{\sqrt{2}}{4}\beta + \tfrac{5}{6}\tfrac{\sqrt{2}}{8}\beta^2 \\ &\geq \tfrac{\sqrt{2}}{16}\alpha + \tfrac{\sqrt{2}}{256}\alpha^2 + \tfrac{\sqrt{2}}{16}(\alpha+\beta) + \tfrac{\sqrt{2}}{256}(\alpha^2+\beta^2) + \tfrac{\beta}{8}. \end{aligned}$$

Consequently we get that

$$\begin{aligned} P + \varepsilon &> 2 + 2\sqrt{2} + \tfrac{\beta}{8} + \tfrac{\sqrt{2}}{16}(\alpha+\beta) + \tfrac{\sqrt{2}}{256}(\alpha^2+\beta^2) + \tfrac{\sqrt{2}}{16}\alpha + \tfrac{\sqrt{2}}{256}\alpha^2 \\ &\geq 2 + \tfrac{\beta}{8} + \sqrt{\left(1 + \tfrac{\alpha}{8}\right)^2 + \left(1 + \tfrac{\beta}{8}\right)^2} + \sqrt{1 + \left(1 + \tfrac{\alpha}{8}\right)^2} \geq S_{3n} \end{aligned}$$

and therefore the map T with $x \in A_1$, $x \in \mathbb{Q} \backslash \{2\}$ is a 3–summing cyclic orbital Meir–Keeler contraction.

It is possible to make the above construction for a uniformly convex Banach space, which is not an Euclidian space, as it is done in the example in [9].

If we consider the map T in the example with the change $T(-1,0) = (2,0)$, then T satisfies all of the condition in Theorem 3.1, except that T is not continuous at $(-1,0)$ and T do not satisfies (3). It is easy to see that $T^3(1,0) \neq (1,0)$.

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