A VARIATIONAL PRINCIPLE AND BEST PROXIMITY POINTS

M. IVANOV, B. ZLATANOV, N. ZLATEVA

Abstract. We generalize Ekeland's Variational Principle for cyclic maps. We present applications of this version of the variational principle for proving of existence and uniqueness of best proximity points for different classes of cyclic maps.

Keywords : Fixed point, cyclical operator, best proximity point, uniformly convex Banach space, variational principle.

Mathematics Subject Classification: 47H10, 54H25, 58E30.

1. Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notion of cyclic maps [10]. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Interesting application of cyclic maps to integro-differential equations is presented in [12]. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of best proximity point is introduced in [7]. This definition is more general than the notion of cyclic maps [10], in sense that if the sets intersect then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [7].

Ekeland formulated a variational principle in [6], which has many applications in different fields of Mathematics. Ekeland's variational principle has many generalizations and applications [2, 5, 11]. There is a close relationship between fixed point theorems and variational principles [6, 3, 4]. Unfortunately there are no results for best proximity points that can be proved with the help of variational principles.

We try to state a variational principle for cyclic maps, which can be applied for proving the existence of best proximity points for different classes of cyclic maps.

2. Preliminary results

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X, ρ) be a metric space. Define a distance between two subset $A, B \subset X$ by $\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. We will use the notation $d = \operatorname{dist}(A, B)$.

Let $A, B \in X$ be nonempty subsets of a metric space (X, ρ). The map $T : A \bigcup B \to A \bigcup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. By f we will denote the function $f : A \cup B \to \mathbb{R}$, which is defined with $f(x) = \rho(x, Tx)$. A point $\xi \in A$ is called a best proximity point of the cyclic map T in A if $f(\xi) = \rho(\xi, T\xi) = \operatorname{dist}(A, B) = d$.

When we investigate Banach space $(X, \|\cdot\|)$ we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$ i.e. $\rho(x, y) = \|x - y\|$.

Definition 2.1. *([5], p. 61) The norm $\|\cdot\|$ on a Banach space X is said to be uniformly convex if $\lim_{n\to\infty}\|x_n-y_n\|=0$ whenever $\|x_n\|=\|y_n\|=1$, $n\in\mathbb{N}$ are such that $\lim_{n\to\infty}\|x_n+y_n\|=2$.*

We will use the following two lemmas, proved in [7].

Lemma 2.2. *([7]) Let A be a nonempty closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty}\|z_n-y_n\|=\operatorname{dist}(A,B)$;
2) for every $\varepsilon>0$ there exists $N_0\in\mathbb{N}$, such that for all $m>n\geq N_0$, $\|x_n-y_n\|\leq\operatorname{dist}(A,B)+\varepsilon$,
then for every $\varepsilon>0$, there exists $N_1\in\mathbb{N}$, such that for all $m>n>N_1$, holds $\|x_m-z_n\|\leq\varepsilon$.

Lemma 2.3. *([7]) Let A be a nonempty closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*
1) $\lim_{n\to\infty}\|x_n-y_n\|=\operatorname{dist}(A,B)$;
2) $\lim_{n\to\infty}\|z_n-y_n\|=\operatorname{dist}(A,B)$;
then $\lim_{n\to\infty}\|x_n-z_n\|=0$.

Definition 2.4. *([5], p. 42) We say that the Banach space $(X,\|\cdot\|)$ is strictly convex if $x=y$ whenever $x,y\in X$ are such that $\|x\|=\|y\|=1$ and $\|x+y\|=2$.*

Let us mention the well known fact, that any uniformly convex Banach space is strictly convex ([5], p.61).

Lemma 2.5. *([15]) Let A, B be closed subsets of a strictly convex Banach space $(X,\|\cdot\|)$, such that $\operatorname{dist}(A,B)>0$ and let A be convex. If $x,z\in A$ and $y\in B$ be such that $\|x-y\|=\|z-y\|=\operatorname{dist}(A,B)$, then $x=z$.*

3. Main result

Theorem 3.1. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space and $A,B\subset X$ be closed, convex sets, $T:A\cup B\to A\cup B$ be a cyclic map, such that $f(x)=\|x-Tx\|$ is lower semi continuous function and let there exists $k\in(0,1)$, such that there holds the inequality*

$$f(Tx)-d\leq k(f(x)-d)$$

for every $x\in A\cup B$.

Then for every $\varepsilon>0$ there exists $v\in A$, such that

$$f(v)\leq\inf\{f(u):u\in A\}+\varepsilon \tag{3.1}$$

and for every $w\in B$ there holds the inequality

$$f(v)\leq f(w)+\varepsilon(\|v-w\|-d). \tag{3.2}$$

4. Auxiliary results

Lemma 4.1. *Let (X,ρ) be a metric space, $A,B\subset X$ be subsets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality*

$$f(Tx)-d\le k(f(x)-d) \tag{4.1}$$

for every $x\in A\cup B$, where $f(x)=\rho(x,Tx)$.

Then there holds the inequality $f(T^nx)-d\le k^n(f(x)-d)$.

Proof. By applying n–times (4.1) we get the inequality

$$f(T^nx)-d\le k(f(T^{n-1}x)-d)\le\cdots\le k^n(f(x)-d).$$

□

Lemma 4.2. *Let (X,ρ) be a metric space, $A,B\subset X$ be subsets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality*

$$f(Tx)-d\le k(f(x)-d)$$

for every $x\in A\cup B$, where $f(x)=\rho(x,Tx)$.

Then $\lim_{n\to\infty}f(T^nx)=d$.

Proof. By Lemma 4.1 we have the inequality

$$0\le\lim_{n\to\infty}(f(T^nx)-d)\le\lim_{n\to\infty}k^n(f(x)-d)=0.$$

Hence we get $\lim_{n\to\infty}f(T^nx)=d$. □

Lemma 4.3. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space, $A,B\subset X$ be closed, convex sets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality $f(Tx)-d\le k(f(x)-d)$ for every $x\in A\cup B$, where $f(x)=\|x-Tx\|$.*

Then $\lim_{n\to\infty}\|T^{2n}x-T^{2n+2}x\|=0$ and $\lim_{n\to\infty}\|T^{2n+1}x-T^{2n+3}x\|=0$.

Proof. By Lemma 4.2 we have the equalities $\lim_{n\to\infty}f(T^{2n}x)=\lim_{n\to\infty}\|T^{2n}x-T^{2n+1}x\|=d$ and $\lim_{n\to\infty}f(T^{2n+1}x)=\lim_{n\to\infty}\|T^{2n+2}x-T^{2n+1}x\|=d$. According to Lemma 2.3 it follows that $\lim_{n\to\infty}\|T^{2n}x-T^{2n+2}x\|=0$.

The proof of the second equality is similar. □

Lemma 4.4. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space, $A,B\subset X$ be closed, convex sets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality $f(Tx)-d\le k(f(x)-d)$ for every $x\in A\cup B$, where $f(x)=\|x-Tx\|$.*

Then there holds the inequality

$$\left\|x-T^{2n+1}x\right\|-d\le\frac{1-k^{2n}}{1-k^2}\|x-T^2x\|+k^{2n}(f(x)-d). \tag{4.2}$$

Proof. We will prove Lemma 4.4 by induction.

I) Let $n=1$. Form Lemma 4.1 it follows that

$$\begin{aligned}\|x-T^3x\|-d &\le \|x-T^2x\|+\|T^2x-T^3x\|-d\\ &= \|x-T^2x\|+f(T^2x)-d\le\|x-T^2x\|+k^2(f(x)-d).\end{aligned}$$

and therefore (4.2) holds true for $n=1$.

II) Let suppose that (4.2) holds true for $n=p$.

III) We will prove that (4.2) holds true for $n=p+1$. Indeed

$$\begin{aligned}\left\|x-T^{2(p+1)+1}x\right\|-d &\leq \left\|x-T^2x\right\|+\left\|T^2x-T^{2(p+1)+1}x\right\|-d\\ &\leq \|x-T^2x\|+k^2(\|x-T^{2p+1}x\|-d)\\ &\leq \|x-T^2x\|+k^2\left(\frac{1-k^{2p}}{1-k^2}\|x-T^2x\|+k^{2p}(f(x)-d)\right)\\ &= \frac{1-k^{2(p+1)}}{1-k^2}\|x-T^2x\|+k^{2(p+1)}(f(x)-d).\end{aligned}$$

□

Lemma 4.5. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space, $A,B\subset X$ be closed, convex sets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality $f(Tx)-d\leq k(f(x)-d)$ for every $x\in A\cup B$, where $f(x)=\|x-Tx\|$. Then*

a) For every $\varepsilon>0$ there exists $N_0\in\mathbb{N}$, such that for every $m\geq n\geq N_0$ there holds the inequality $\|T^{2n}x-T^{2m+1}x\|<d+\varepsilon$.

b) For every $\varepsilon>0$ there exists $N_0\in\mathbb{N}$, such that for every $m>n\geq N_0$ there holds the inequality $\|T^{2m}x-T^{2n+1}x\|<d+\varepsilon$.

Proof. a) Put $v=T^{2n}x$. Then $T^{2m+1}x=T^{2(m-n)+1}v$. From Lemma 4.4 we have that

$$\begin{aligned}\|T^{2n}x-T^{2m+1}x\|-d &= \|v-T^{2(m-n)+1}v\|-d\\ &\leq \frac{1-k^{2(m-n)}}{1-k^2}\|v-T^2v\|+k^{2(m-n)}(f(v)-d)\\ &\leq \frac{1}{1-k^2}\|v-T^2v\|+k^{2(m-n)}(f(v)-d)\\ &= \frac{1}{1-k^2}\|T^{2n}x-T^{2n+2}x\|+k^{2(m-n)}(\|T^{2n}x-T^{2n+1}x\|-d).\end{aligned}$$

From Lemma 4.3 and Lemma 4.2 it follows that there exists $N_0\in\mathbb{N}$, such that for every $n\geq N_0$ there hold the inequalities $\|T^{2n}x-T^{2n+2}x\|<\frac{(1-k^2)\varepsilon}{2}$ and $\|T^{2n}x-T^{2n+1}x\|-d<\frac{\varepsilon}{2}$. Consequently for every $m\geq n\geq N_0$ there holds the inequality

$$\|T^{2n}x-T^{2m+1}x\|-d<\varepsilon.$$

b) Put $v=T^{2n+1}x$. Then $T^{2m}x=T^{2(m-n)-1}v$. From Lemma 4.4 we have that

$$\begin{aligned}\|T^{2n+1}x-T^{2m}x\|-d &= \|v-T^{2(m-n)-1}v\|-d\\ &\leq \frac{1-k^{2(m-n-1)}}{1-k^2}\|v-T^2v\|+k^{2(m-n-1)}(f(v)-d)\\ &\leq \frac{1}{1-k^2}\|v-T^2v\|+k^{2(m-n-1)}(f(v)-d)\\ &= \frac{1}{1-k^2}\|T^{2n+1}x-T^{2n+3}x\|+k^{2(m-n-1)}(\|T^{2n+1}x-T^{2n+2}x\|-d).\end{aligned}$$

From Lemma 4.3 and Lemma 4.2 it follows that there exists $N_0\in\mathbb{N}$, such that for every $n\geq N_0$ there hold the inequalities $\|T^{2n+1}x-T^{2n+3}x\|<\frac{(1-k^2)\varepsilon}{2}$ and $\|T^{2n+1}x-T^{2n+2}x\|-d<\frac{\varepsilon}{2}$. Consequently for every $m\geq n\geq N_0$ there holds the inequality

$$\|T^{2n+1}x-T^{2m}x\|-d<\varepsilon.$$

□

Lemma 4.6. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space, $A,B\subset X$ be closed, convex sets, $T:A\cup B\to A\cup B$ be a cyclic map and let there exists $k\in(0,1)$, such that there holds the inequality $f(Tx)-d\le k(f(x)-d)$ for every $x\in A\cup B$.*

Then for every $x\in A$ the sequences $\{T^{2n}x\}_{n=1}^{\infty}$ and $\{T^{2n+1}x\}_{n=1}^{\infty}$ are Cauchy sequences.

Proof. By Lemma 4.2 we have that $\lim_{n\to\infty}\|T^{2n}x-T^{2n+1}x\|=d$. By Lemma 4.5 b) we have that for every $\varepsilon>0$ there exists $N_0\in\mathbb{N}$, such that there holds the inequality

$$\|T^{2m}x-T^{2n+1}x\|<d+\varepsilon.$$

for every $m\ge n\ge N_0$. According to Lemma 2.2 there exists $N_1\in\mathbb{N}$, such that the inequality $\|T^{2m}x-T^{2n}x\|<\varepsilon$ holds for every $m>n\ge N_1$.

The proof that the sequence $\{T^{2n+1}x\}_{n=1}^{\infty}$ is a Cauchy sequence is similar. □

5. Proof of main result

From Lemma 4.2 it follows that $\inf\{f(u):u\in A\}=d$. Let $\varepsilon>0$ be arbitrary. We choose arbitrary $x\in A$. Put $u_0=x$ and $u_n=T^nx=Tu_{n-1}$, $n\in\mathbb{N}$. From Lemma 4.5 it follows that for every $\varepsilon>0$ there exists $N_0\in\mathbb{N}$, such that the inequality $f(u_{2n})=f(T^{2n}x)\le d+\varepsilon=\inf\{f(u):u\in A\}+\varepsilon$ holds for every $n\ge N_0$.

There are two cases:

I) There exists $n_0\ge N_0$, such that for every $w\in B$ there holds the inequality

$$f(w)\ge f(u_{2n_0})-\varepsilon\left(\|u_{2n_0}-w\|-d\right).$$

In this case we put $v=u_{2n_0}$ and the proof is finished.

II) For every $n\ge N_0$ there exists $w_n\in B$, such that

$$f(w_n)<f(u_{2n})-\varepsilon\left(\|u_{2n}-w\|-d\right).$$

Then the sets $S_n=\{w\in B: f(w)<f(u_{2n})-\varepsilon(\|u_{2n}-w\|-d)\}$ are not empty for $n\ge N_0$. By Lemma 4.6 the sequence $\{u_{2n}\}_{n=1}^{\infty}$ is a Cauchy sequence and thus it is convergent to some $v\in A$.

We will show that for every $w\in B$ there holds the inequality

$$f(v)\le f(w)+\varepsilon(\|v-w\|-d).$$

Let us suppose the contrary, i.e there is $w\in B$, such that there holds the inequality

$$f(w)<f(v)-\varepsilon(\|v-w\|-d). \tag{5.1}$$

First we will show that if there is $w\in B$, that satisfies (5.1), then $\|v-w\|>d$. If not then from the lower semi continuity of f and Lemma 4.2 we get

$$f(w)<f(v)-\varepsilon(\|v-w\|-d)\le f(v)\le\lim_{n\to\infty}f(T^{2n}x)=d,$$

which is a contradiction, because $f(w)=\|w-Tw\|\ge d$ for any $w\in B$.

Thus if there exists $w\in B$, that satisfies (5.1), then $\|v-w\|>d$. By $\lim_{n\to\infty}u_{2n}=v$ and the lower semi continuity of f we obtain the inequality

$$f(w)<f(v)-\varepsilon(\|v-w\|-d)\le\lim_{n\to\infty}(f(u_{2n})-\varepsilon(\|u_{2n}-w\|-d)). \tag{5.2}$$

We claim that there is $N_1\in\mathbb{N}$, such that for every $n\ge N_1$ there holds the inequality

$$f(w)<f(u_{2n})-\varepsilon(\|u_{2n}-w\|-d). \tag{5.3}$$

If (5.3) does not hold, then there is a subsequence of naturals $\{n_k\}_{k=1}^{\infty}$, such that

$$f(w) \geq f(u_{2n_k}) - \varepsilon(\|u_{2n_k} - w\| - d)$$

and consequently by the lower semi continuity of f the inequality

$$f(w) \geq \lim_{k\to\infty} (f(u_{2n_k}) - \varepsilon(\|u_{2n_k} - w\| - d)) \geq f(v) - \varepsilon(\|v - w\| - d)$$

should hold, which is a contradiction with (5.2). Thus there is $N_1 \in \mathbb{N}$, such that (5.3) holds for every $n \geq N_1$. Therefore $w \in S_n$ for any $n \geq \max\{N_0, N_1\}$.

From Lemma 4.2, the lower semi continuity of f and the construction of the sequence $\{u_n\}_{n=1}^{\infty}$ it follows that

$$f(v) \leq \lim_{n\to\infty} f(u_{2n}) = d \leq \inf_{x\in S_n} f(x) \leq f(w),$$

which is a contradiction with (5.1), because $\|v - w\| > d$. Consequently for every $w \in B$ there holds the inequality

$$f(v) \leq f(w) + \varepsilon(\|v - w\| - d).$$

6. Applications

Theorem 6.1. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $k \in (0, 1)$, such that there holds the inequality*

$$\|Tx - T^2x\| \leq k\|x - Tx\| + (1 - k)d \tag{6.1}$$

for every $x \in A$.

Then there is a best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. Put $F(x) = f(x) - d = \|x - Tx\| - d$. From (6.1) it follows the inequality $F(Tx) \leq kF(x)$. Let $\varepsilon \in (0, 1 - k)$ be arbitrary chosen. By Theorem 3.1 it follows that there exists $v \in A$, such that the inequality $f(v) \leq f(w)+\varepsilon(\|v-w\|-d)$ holds for every $w \in B$ and $f(v) \leq \inf\{f(u) : x \in A\}+\varepsilon$. Let us choose $w = Tv \in B$. Then from the inequality

$$F(v) = f(v) - d \leq f(Tv) - d + \varepsilon(\|v - Tv\| - d) = F(Tv) + \varepsilon F(v) \leq kF(v) + \varepsilon F(v)$$

we get that $(1 - k - \varepsilon)F(v) \leq 0$. By the choice of $\varepsilon \in (0, 1 - k)$ it follows that $F(v) = 0$ and thus v is a best proximity point of the cyclic map T in A.

We will show that $T^2v = v$, where $v \in A$ is a best proximity point for the map T in A. By (6.1) it follows that

$$\|Tv - T^2v\| \leq k\|v - Tv\| + (1 - k)d = d.$$

By the uniform convexity of $(X, \|\cdot\|)$ and the choice of v, such that $\|Tv - v\| = d$ it follows from Lemma 2.5 that $T^2v = v$. □

Let us mention that in order to apply the variational principle we need to impose an additional condition: the function $f(x) = \|x - Tx\|$ to be lower semi continuous. That is why the next Theorems are weaker variants of the original ones.

Theorem 6.2. *(Cyclic contraction [7]) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A\cup B \to A\cup B$ be a cyclic map, $f(x) = \|x-Tx\|$ be lower semi continuous function and let there exists $k \in (0, 1)$, such that there holds the inequality*

$$\|Tx - Ty\| \leq k\|x - y\| + (1 - k)d \tag{6.2}$$

for every $x \in A$, $y \in B$.

Then there is a unique best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. If we put $y = Tx$ in (6.2) then T satisfies (6.1) and we can apply Theorem 6.1.

It remains to show that v is a unique best proximity point. Let us suppose the contrary, i.e. there exists $u \neq v$, such that $\|u - Tu\| = d$. We can prove in a similar fashion, as it is done in Theorem 6.1, that $T^2u = u$. Thus we can write the chain of inequalities

$$\begin{aligned} \|u - Tv\| - d &= \|T^2u - Tv\| - d \leq k(\|Tu - v\| - d) \\ &= k(\|Tu - T^2v\| - d) \leq k^2(\|u - Tv\| - d). \end{aligned} \tag{6.3}$$

From (6.3) it follows that $\|u - Tv\| = d$. Using the fact that $\|v - Tv\| = d$, the uniform convexity of X and Lemma 2.5 it follows that $u = v$. □

Theorem 6.3. *(Reich type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $a, b, c \geq 0$, $0 \leq a+b+c < 1$, such that there holds the inequality*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\| + (1 - a - b - c)d \tag{6.4}$$

for every $x \in A$, $y \in B$.

Then there is a unique best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. If we put $y = Tx$ in (6.4) we get

$$\begin{aligned} \|Tx - T^2x\| &\leq a\|x - Tx\| + b\|Tx - x\| + c\|T^2x - Tx\| + (1 - a - b - c)d \\ &\leq c\|T^2x - Tx\| + (a + b)\|Tx - x\| + (1 - a - b - c)d. \end{aligned}$$

Thus

$$\begin{aligned} \|Tx - T^2x\| &\leq \frac{a + b}{1 - c}\|Tx - x\| + \frac{1 - c - (a + b)}{1 - c}d \\ &= \frac{a + b}{1 - c}\|Tx - x\| + \left(1 - \frac{a + b}{1 - c}\right) d. \end{aligned} \tag{6.5}$$

From $0 \leq a + b + c < 1$ it follows that $\frac{a+b}{1-c} \in (0, 1)$ and therefore T satisfies (6.1) and we can apply Theorem 6.1.

It remains to show that v is a unique best proximity point of T in A. Let us suppose the contrary, i.e. there exists $u \neq v$, such that $\|u - Tu\| = d$. We can prove in a similar fashion, as it is done in Theorem 6.1, that $T^2u = u$. Thus we can write the chain of inequalities

$$\begin{aligned} \|u - Tv\| - d &= \|T^2u - Tv\| - d \\ &\leq a(\|Tu - v\| - d) + b(\|T^2u - Tu\| - d) + c(\|Tv - v\| - d) \\ &= a(\|Tu - v\| - d) + b(\|u - Tu\| - d \\ &= a(\|Tu - v\| - d) = a(\|Tu - T^2v\| - d) \\ &\leq a^2(\|u - Tv\| - d) + ab(\|Tu - u\| - d) + ac(\|T^2v - Tv\| - d) \\ &= a^2(\|u - Tv\| - d) + ac(\|v - Tv\| - d) = a^2(\|u - Tv\| - d). \end{aligned} \tag{6.6}$$

From (6.6) it follows that $\|u - Tv\| = d$. Using the fact that $\|v - Tv\| = d$, the uniform convexity of X and Lemma 2.5 it follows that $u = v$. □

Theorem 6.4. *(Kannan type cyclic contraction [13]) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $k \in (0, 1/2)$, such that there holds the inequality*

$$\|Tx - Ty\| \le k(\|Tx - x\| + \|Ty - y\|) + (1 - 2k)d \tag{6.7}$$

for every $x \in A$, $y \in B$.

Then there is a unique best proximity point $u \in A$ of the cyclic map T in A, such that $T^2u = u$.

Proof. Kannan type cyclic contraction is a particular case of Reich type cyclic contraction with $a = 0$ and $b = c = k$. □

Various types of contractive maps can be found in [1, 14]. We have tried to extend the results on best proximity points for some classical contractive maps.

Theorem 6.5. *(Ciric type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $k \in (0, 1)$, such that there holds the inequality*

$$\|Tx - Ty\| \le kM(x, y) + (1 - k)d \tag{6.8}$$

for every $x \in A$, $y \in B$, where $M(x, y) = \max\{\|x - y\|, \|Tx - x\|, \|Ty - y\|\}$.

Then there is a unique best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. If we put $y = Tx$ in (6.8) we get the inequality

$$\|Tx - T^2x\| \le kM(x, Tx) + (1 - k)d \le k \max\{\|Tx - x\|, \|T^2x - Tx\|\} + (1 - k)d.$$

If there holds $M(x, Tx) = \|T^2x - Tx\|$ we get that $\|Tx - T^2x\| = d$ and thus

$$\|Tx - T^2x\| = d \le k\|x - Tx\| + (1 - k)d. \tag{6.9}$$

If there holds $M(x, Tx) = \|x - Tx\|$ we get that

$$\|Tx - T^2x\| \le k\|Tx - x\| + (1 - k)d. \tag{6.10}$$

Therefore from (6.9) and (6.10) it follows that T satisfies (6.1) and we can apply Theorem 6.1.

To prove that $T^2v = v$, where v is a best proximity point of T in A we use the inequality

$$\begin{aligned} \|Tv - T^2v\| - d &\le k(M(v, Tv) - d) \\ &= k(\max\{\|Tv - v\|, \|T^2v - Tv\|\} - d) \\ &= k(\|T^2v - Tv\| - d). \end{aligned}$$

to obtain that $\|T^2v - Tv\| = d$. By the uniform convexity of $(X, \|\cdot\|)$ and the choice of v, such that $\|Tv - v\| = d$ it follows from Lemma 2.5 that $T^2v = v$.

It remains to show that v is a unique best proximity point of T in A. Let us suppose the contrary, i.e. there exists $u \ne v$, such that $\|u - Tu\| = d$. We can prove in a similar fashion, as it is done in Theorem 6.1, that $T^2u = u$. Thus we can write the inequality

$$\|u - Tv\| - d = \|T^2u - Tv\| - d \le k(M(Tu, v) - d) \tag{6.11}$$

There are two cases: I) $M(Tu,v) = \|T^2u - Tu\| = \|u - Tu\| = d$ or $M(Tu,v) = \|Tv - v\| = d$; II) $M(Tu,v) = \|Tu - v\|$.

I) If there holds $M(Tu,v) = \|T^2u - Tu\|$ or $M(Tu,v) = \|Tv - v\|$ then from (6.11) we get the inequality

$$\|u - Tv\| - d \leq k(M(Tu,v) - d) = 0 \tag{6.12}$$

and consequently $\|u - Tv\| = d$. Using the fact that $\|v - Tv\| = d$, the uniform convexity of X and Lemma 2.5 it follows that $u = v$.

II) If there holds $M(Tu,v) = \|Tu - v\|$ then from (6.11) we can write the inequality

$$\|u - Tv\| - d \leq k(\|Tu - v\| - d) = k(\|Tu - T^2v\| - d) \leq k^2(M(u,Tv) - d). \tag{6.13}$$

There are two subcases: II.1) $M(u,Tv) = \|Tu - u\| = d$ or $M(u,Tv) = \|T^2v - Tv\| = \|v - Tv\| = d$; II.2) $M(u,Tv) = \|u - Tv\|$.

II.1) If there holds $M(u,Tv) = \|Tu - u\|$ or $M(u,Tv) = \|T^2v - Tv\|$ then from (6.13) we get the inequality

$$\|u - Tv\| - d \leq k^2(M(u,Tv) - d) = 0 \tag{6.14}$$

and consequently $\|u - Tv\| = d$. Using the fact that $\|v - Tv\| = d$, the uniform convexity of X and Lemma 2.5 it follows that $u = v$.

II.2) If there holds $M(u,Tv) = \|u - Tv\|$ then from (6.13) we get the inequality

$$\|u - Tv\| - d \leq k^2(\|u - Tv\| - d). \tag{6.15}$$

From (6.15) it follows that $\|u - Tv\| = d$. Using the fact that $\|v - Tv\| = d$, the uniform convexity of X and Lemma 2.5 it follows that $u = v$. □

Theorem 6.6. *(Hardy and Rogers type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $p, q \geq 0$, $0 \leq p + 2q < 1$, such that there holds the inequality*

$$\|Tx - Ty\| \leq p\|x - y\| + q\|Tx - y\| + q\|Ty - x\| + (1 - p - 2q)d \tag{6.16}$$

for every $x \in A$, $y \in B$.

Then there is a best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. If we put $y = Tx$ in (6.16) we get the inequality

$$\begin{aligned} \|Tx - T^2x\| &\leq p\|x - Tx\| + q\|T^2x - x\| + (1 - p - 2q)d \\ &\leq p\|Tx - x\| + q\|T^2x - Tx\| + q\|Tx - x\| + (1 - p - 2q)d. \end{aligned}$$

Thus

$$\|Tx - T^2x\| \leq \frac{p+q}{1-q}\|Tx - x\| + \frac{1-p-2q}{1-q}d = \frac{p+q}{1-q}\|Tx - x\| + \left(1 - \frac{p+q}{1-q}\right)d. \tag{6.17}$$

From $0 \leq p + 2q < 1$ it follows that $\frac{p+q}{1-q} \in (0,1)$ and therefore T satisfies (6.1) and we can apply Theorem 6.1. □

Theorem 6.7. *(Chatterjee type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exists $k \in (0, 1/2)$, such that there holds the inequality*

$$\|Tx - Ty\| \leq k(\|Tx - y\| + \|Ty - x\|) + (1 - 2k)d \tag{6.18}$$

for every $x \in A$, $y \in B$.

Then there is a best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. Chatterjee type cyclic contraction is a particular case of Hardy and Rogers type cyclic contraction with $p = 0$ and $q = k$. □

Theorem 6.8. *(Zamfirescu type cyclic contraction) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $A, B \subset X$ be closed, convex sets, $T : A \cup B \to A \cup B$ be a cyclic map, $f(x) = \|x - Tx\|$ be lower semi continuous function and let there exist $\alpha \in (0, 1)$ and $\beta, \gamma \in (0, 1/2)$, such that for each $x \in A$, $y \in B$, at least one of the following is true:*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + (1 - \alpha)d; \tag{6.19}$$

$$\|Tx - Ty\| \leq \beta(\|Tx - x\| + \|Ty - y\|) + (1 - 2\beta)d; \tag{6.20}$$

$$\|Tx - Ty\| \leq \gamma(\|Tx - y\| + \|Ty - x\|) + (1 - 2\gamma)d. \tag{6.21}$$

Then there is a best proximity point $u \in A$ of T in A, such that $T^2u = u$.

Proof. If (6.20) holds, then from (6.5) with $a = 0$ and $b = c = \beta$ we get the inequality $\|Tx - T^2x\| - d \leq \frac{\beta}{1-\beta}(\|x - Tx\| - d)$. If (6.21) holds, then from (6.17) with $p = 0$ and $q = \gamma$ we get $\|Tx - T^2x\| - d \leq \frac{\gamma}{1-\gamma}(\|x - Tx\| - d)$. Therefore the following inequality

$$\|Tx - T^2x\| - d \leq \lambda(\|x - Tx\| - d) \tag{6.22}$$

holds true, where $\lambda := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$.

From $0 \leq \lambda < 1$ it follows that T satisfies (6.1) and we can apply Theorem 6.1. □

We would like to pose an open question if the best proximity point for Hardy and Rogers, Chatterjee or Zamfirescu type cyclic contraction is unique.

References

[1] Berinde, V.: Iterative Approximation of Fixed Points, Springer, Berlin Heidelberg New York, 2007

[2] Borwein, J., Zhu, Q.: Techniques of Variational Analysis, Springer, Berlin Heidelberg NewYork Hong Kong London Milan Paris Tokyo, 2005

[3] Caristi, J.: Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Amer. Math. Soc.*, **215**, 241-251 (1976)

[4] Daffer, P., Kaneko, H., Li, W.: Variational principle and fixed points. Set Valued Mappings With Applications in Nonlinear Analysis, Taylor and Francis, London, 2002

[5] Deville, R., Godefroy, G., Zizler, V.: Smoothness and renormings in Banach spaces Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman Scientific and Technical, Harlow, 1993

[6] Ekeland, I.: Nonconvex Minimization Problems, *Bull. Amer. Math. Soc.*, **1**(3), 443–474 (1979)

[7] Eldred, A., Veeramani, P.: Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323**(2), 1001–1006 (2006)

[8] Kadelburg, Z., Radenović, S.: A note on some recent best proximity point results for non–self mappings, *Gulf Journal of Mathematics*, **1**, 36–41 (2013)

[9] Karpagam, S., Sushama Agrawal.: Best proximity point theorems for p-Cyclic Meir-Keeler contractions. *Fixed Point Theory Appl.*, **2009**, Article ID 197308, (2009), 9 pages. DOI:10.1155/2009/197308

[10] Kirk, W., Srinivasan, P., Veeramani, P.: Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**, 79–189 (2003)

[11] Meghea, I.: Ekeland Variational Principle with Generalizations and Variants, Old City Pubishing, Philadelphia, 2009

[12] Nashine, H., Kadelburg, Z.: Weaker Cyclic (φ, ϕ)–contractive mappings with an application to integro–differential equations *Nonlinear Anal. Model. Control*, **18**(4), 427–443 (2013)

[13] Petric, M.: Best proximity point theorems for weak cyclic Kannan contractions, *Filomat*, **25**(1), 145-154 (2011)

[14] Rhoades, B.: A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226**(2), 257–290 (1977)

[15] Zlatanov, B.: Best Proximity Points for p–Summing Cyclic Orbital Meir–Keeler Contractions, *Nonlinear Anal. Model. Control*, (to appear)

M. Ivanov: Faculty of Mathematics and Informatics,
Sofia University "St. Kliment Ohridski",
5, James Bourchier blvd., 1164 Sofia, Bulgaria.
E-mail :milen@fmi.uni-sofia.bg

B. Zlatanov: Faculty of Mathematics and Informatics,
Plovdiv University "Paisii Hilendarski",
24, Tsar Assen str., 4000 Plovdiv, Bulgaria.
E-mail :bzlatanov@gmail.com

N. Zlateva Faculty of Mathematics and Informatics,
Sofia University "St. Kliment Ohridski",
5, James Bourchier blvd., 1164 Sofia, Bulgaria.
E-mail :zlateva@fmi.uni-sofia.bg