Upper and lower estimates in weighted Orlicz sequence spaces and Lorentz–Orlicz sequence spaces [1]

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ABSTRACT:We study the existence of lower and upper p–estimates in weighted Orlicz sequence spaces and Lorentz–Orlicz sequence spaces. We also find upper estimates for the unit vector basis in these spaces. We give some applications to weak sequential continuity of polynomials.

1 Introduction

The existence of ℓ_p–estimates in the sequences in a Banach space is of great interest in the study of the structure of space. It is also relevant in some questions of non linear analysis such as the behaviour of polynomials [1], [7], [10], [20], [24]. The upper and lower ℓ_p–estimates in sequences are of great interest when studding reflexivity of the space of polynomials [2], [6], [18] and the problems of smoothness in Banach spaces [10], [11], [17], [18] in the sense of existence of real bump functions with higher order of differentiability [10], [4]. The existence of these estimates and the behavior of polynomials has been studied in [10], [9].

We use the standard Banach space terminology from [16].

We will begin with the following well known definition:

Definition 1.1 *Let X be a Banach space and $1 < p, q \le \infty$. A sequence $\{x_k\}_{k=1}^\infty$ of elements of X is said to have upper p–estimate (respectively a lower q–estimate) if there exists a constant $C > 0$ such that for all scalars $a_1, a_2, \dots, a_n$ and all $n \in \mathbb{N}$ we have*

$$\left\| \sum_{k=1}^n a_i x_i \right\| \le C \left(\sum_{i=1}^n |a_i|^p \right)^{1/p}$$

$$\left(\text{respectively} \quad C \left(\sum_{i=1}^n |a_i|^q \right)^{1/q} \le \left\| \sum_{k=1}^n a_i x_i \right\| \right)$$

Here $(\sum |a_k|^\infty)^{1/\infty} = \max_{k \in \mathbb{N}} |a_k|$.

Definition 1.2 *A Banach space X is said to have property S_p if every weakly null normalized sequence $\{x_k\}_{k=1}^\infty$ has a subsequence $\{x_{k_i}\}_{i=1}^\infty$ with an upper p–estimate [15].*

A Banach space X is said to have property T_p if every weakly null normalized sequence $\{x_k\}_{k=1}^\infty$ has a subsequence $\{x_{k_i}\}_{i=1}^\infty$ with an lower q–estimate [10].

Definition 1.3 *A Banach space X has US_p–property [15] (respectively UT_q–property [10]) if there is an Constant $C > 0$ such that every weakly null normalized sequence $\{x_k\}_{k=1}^\infty$ has a subsequence $\{x_{k_i}\}_{i=1}^\infty$ with an upper p–estimate (respectively lower q–estimate) with a constant C.*

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It is proved in [15] that the properties S_p and US_p coincide. The properties T_q and UT_q are not equivalent, even in reflexive Banach spaces [8].

The following notion was introduced in [12].

Definition 1.4 *Let $1 < p \leq \infty$. A Banach space has Banach–Saks type p (for short BS_p–property) if every weakly null normalized sequence $\{x_k\}_{k=1}^{\infty}$ has a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ so that for some constant $0 < C < \infty$ holds:*

$$\left\| \sum_{i=1}^{n} x_{k_i} \right\| \leq C n^{1/p}$$

for every $n \in \mathbb{N}$. Here $n^{1/\infty} = 1$.

Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. We say that M has the Δ_2–condition at zero, if for some constant $K > 0$ and $t_0 > 0$ it holds $M(2t) \leq KM(t)$ for every $0 < t \leq t_0$.

Throughout this paper M will always denote Orlicz function.

For a positive measure space (Ω, Σ, μ) the Orlicz space $L_M(\mu)$ is defined as the set of all equivalence classes of μ–measurable scalar functions x on Ω such that for some $\lambda > 0$

$$\widetilde{M}(x/\lambda) = \int_{\Omega} M\left(\frac{x(t)}{\lambda}\right) d\mu(t) < \infty$$

Usually the Orlicz space $L_M(\mu)$ is equipped with the so called Luxemburg norm:

$$\|x\|_{\ell_M(w)} = \inf\{\lambda > 0 : \widetilde{M}(x/\lambda) \leq 1\}.$$

An extensive study of Orlicz spaces can be found in [16].

For $\Omega = \mathbb{N}$ and $w = \{w_j\}_{j=1}^{\infty} = \{\mu(j)\}_{j=1}^{\infty}$ we get the weighted Orlicz sequence space $\ell_M(w)$. In this case we have $x \in \ell_M(w)$ iff $\widetilde{M}(x/\lambda) = \sum_{i=1}^{\infty} w_i M(x_i/\lambda) < \infty$ for some $\lambda > 0$ and $\|x\|_{\ell_M(w)} = \inf\{\lambda > 0 : \sum_{i=1}^{\infty} w_i M(x_i/\lambda) \leq 1\}$.

It is well know that the space $\ell_M(w)$ endowed with the Luxemburg norm $\|\cdot\|_{\ell_M(w)}$ is a Banach space. The unit vectors $\{e_i\}_{i=1}^{\infty}$ form an unconditional basis in $\ell_M(w)$.

We denote by $h_M(w)$ the subspace of $\ell_M(w)$ consisting of those sequences $\{x_i\}_{i=1}^{\infty}$, such that $\sum_{i=1}^{\infty} w_i M(\lambda x_i) < \infty$ for every $\lambda > 0$.

By $w \in \Lambda$, we mean that there exists a subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ of w such that

$$\lim_{k\to\infty} w_{j_k} = 0 \text{ and } \sum_{k=1}^{\infty} w_{j_k} = \infty.$$

When $w_j = 1$ for each $j \in \mathbb{N}$, we obtain the usual Orlicz sequence space denoted by ℓ_M.

Some results concerning S_p and T_q properties in Orlicz sequence spaces h_M are obtained in [14] and [8].

Theorem 1 *([14]) Let $1 < p < \infty$, M is an Orlicz function, Let h_M be an Orlicz sequence space not containing ℓ_1. Then the following are equivalent:*
(a) h_M has property BS_p;
(b) the Orlicz function M satisfies:

$$\sup_{0<s,t\leq 1} \frac{M(st)}{M(s)t^p} < \infty;$$

(c) h_M has property S_p.

Theorem 2 *[8] Let $1 < q < \infty$, M is an Orlicz function. Let h_M be an Orlicz sequence space not containing ℓ_1. Then the following are equivalent:*
(a) h_M has property UT_q;
(b) the Orlicz function M satisfies:

$$\inf_{0<s,t\leq 1} \frac{M(st)}{M(s)t^p} > 0.$$

Let $w = \{w_i\}_{i=1}^{\infty}$ be a a positive decreasing sequence such that $w_1 = 1$, $\lim_{i\to\infty} w_i = 0$ and $\lim_{n\to\infty} W(n) = \infty$, where $W(n) = \sum_{i=1}^{n} w_i$ for every $n \in \mathbb{N}$. The Lorentz–Orlicz sequence space $d(w, M)$ consists of all bounded real sequences $x = \{x_n\}_{n=1}^{\infty}$ such that for some $\lambda > 0$ holds $I(\lambda x) < \infty$, where

$$I(x) = \sum_{i=1}^{\infty} w_i M(x_i^*) = \sup\left\{\sum_{i=1}^{\infty} w_i M(x_{\pi(i)}) : \pi \text{ is an injection } \mathbb{N} \to \mathbb{N}\right\},$$

and $x^* = \{x_i^*\}_{i=1}^{\infty}$ is the decreasing rearrangement of $|x| = \{|x_n|\}_{n=1}^{\infty}$. The space $d(w, M)$ equipped with the Luxemburg norm

$$\|x\|_{d(w,M)} = \inf\{\lambda > 0 : I(x/\lambda) \leq 1\} \tag{1}$$

is a Banach space.

Notice that the assumption $\lim_{n\to\infty} W(n) = \infty$ yields that $d(w, M) \hookrightarrow c_0$.

We denote by $d_0(w, M)$ the closure of finitely supported sequences in $d(w, M)$.

The next proposition from [13] shows that the space $d(w, M)$ has much in common with ℓ_M.

Proposition 1.1 *([13]) I) The subspace $d_0(w, M)$ coincides with the set of all sequences $x = \{x_n\}_{n=1}^{\infty}$ such that for every $\lambda > 0$ holds $I(\lambda x) < \infty$. Moreover, the sequence of the unit vectors $\{e_i\}_{i=1}^{\infty}$ is a symmetric basis in $d_0(w, M)$.*
II) The following assertions are equivalent:
i) The Orlicz function M satisfies the Δ_2–condition;
ii) the unit vectors $\{e_i\}_{i=1}^{\infty}$ form a boundedly complete basis in $d_0(w, M)$;
iii) $d_0(w, M) = d(w, M)$;
iv) $d_0(w, M)$ does not contain a closed subspace isomorphic to c_0.

If $M(t) = t^p$, $1 \le p < \infty$, then $d(w,M) = d(w,p)$ is the Lorentz sequence space. If $w_i = 1$ for every $i \in \mathbb{N}$, then $d(w,M) = \ell_M$ is the Orlicz sequence space and $h_M = d_0(w,M)$.

The unit vectors $\{e_i\}_{i=1}^{\infty}$ form a symmetric basis in $d_0(w,M)$.

The symbol e_n will stand for the unit vectors in $h_M(w)$ and $d_0(w,M)$.

A sufficient condition for existence of property S_p in Lorentz sequence spaces $d(w,p)$ is obtained in [3].

Theorem 3 *[3] The Lorenz sequence space $d(w,p)$ has property S_p if $1 < p < \infty$.*

The following fact is well known, (see [14]) but for the sake of completeness we will prove it.

Proposition 1.2 *If a Banach space X has property S_p then X has property BS_p.*

Proof: Let $\{x_k\}_{k=1}^{\infty}$ be a weakly null normalized sequence in X. By the fact that X has property S_p follows that there is a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ such that

$$\left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \le C \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p}$$

for all $n \in \mathbb{N}$ and all sequences $\{a_i\}_{i=1}^{n}$.

Let $a_i = 1/n^{1/p}$. Then $\sum_{i=1}^{n} |a_i|^p = \sum_{i=1}^{n} \left(\frac{1}{n^{1/p}} \right)^p = 1$. Thus $\left\| \sum_{i=1}^{n} \frac{1}{n^{1/p}} x_{k_i} \right\| \le C$ or equivalently $\|\sum_{i=1}^{n} x_{k_i}\| \le C n^{1/p}$. □

It follows from Elton's c_0–theorem [5] that BS_∞–property implies S_∞–property. Moreover both properties are equivalent to the hereditary Danford–Pettis property. For $1 < p < \infty$, however, both properties are not equivalent: the Lorentz sequence space $d(w,1)$, where $W(n) = n^{1/p}$ for all $n \in \mathbb{N}$, has property BS_p, while failing property S_p. It is proved in [21] that property BS_p implies $S_{p-\varepsilon}$ for any $\varepsilon > 0$.

If $\ell_1 \hookrightarrow h_M$, then $\alpha_M = 1$. It shown [22] that in this case either h_M has BS_∞–property and thus S_∞, or h_M fails BS_p–property and thus S_p for all $p > 1$. Consequently, property BS_p and S_p are equivalent for all Orlicz sequence spaces.

We do not know of an example of a Banach space, not containing ℓ_1, which has property BS_p and fails property S_p.

Definition 1.5 *Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in a Banach space X. A sequence of non–zero vectors $\{u_j\}_{j=1}^{\infty}$ in X of the form $u_j = \sum_{n=p_j+1}^{p_{j+1}} a_n x_n$ with $\{a_n\}_{n=1}^{\infty}$ scalars and $p_1 < p_2 \ldots < p_j < \ldots$ an increasing sequence of integers, is called a block basic sequence of $\{x_n\}_{n=1}^{\infty}$.*

Definition 1.6 *Two bases, $\{x_n\}_{n=1}^{\infty}$ of X and $\{y_n\}_{n=1}^{\infty}$ of Y, are called equivalent provided a series $\sum_{n=1}^{\infty} a_n x_n$ converges iff $\sum_{n=1}^{\infty} a_n y_n$ converges.*

The next Proposition is in fact an easy consequence results in ([16], p.7). As it is not stated there in terms of normalized weakly null sequences and normalized block basic sequences and just for the sake of completeness we will prove it.

Proposition 1.3 *Let $\{x_i\}_{i=1}^{\infty}$ be a basis of a Banach space X with a basis constant K. Let $y^{(k)} = \sum_{i=1}^{\infty} a_i^{(k)} x_i$, $k \in \mathbb{N}$ be a weakly null normalized sequence in X. Then there exist a subsequence $\{y^{(k_n)}\}_{n=1}^{\infty}$ of $\{y^{(k)}\}_{k=1}^{\infty}$, which is equivalent to a normalized block basic sequence of $\{x_i\}_{i=1}^{\infty}$.*

Proof: We will construct the block basic sequence of $\{x_i\}_{i=1}^{\infty}$ inductively.
1) Let take $y^{(k_1)} = y^{(1)}$. Find $p_1 \in \mathbb{N}$ such that $\left\|\sum_{i=p_1+1}^{\infty} a_i^{(1)} x_i\right\| \leq \dfrac{1}{2.4.K}$. Put $u_1 = \sum_{i=1}^{p_1} a_i^{(1)} x_i$ and $v_1 = u_1/\|u_1\|$. Obviously $\|v_1\| = 1$ and

$$1 \geq \|u_1\| = \left\|\sum_{i=1}^{\infty} a_i^{(1)} x_i - \sum_{i=p_1+1}^{\infty} a_i^{(1)} x_i\right\| \geq \left\|\sum_{i=1}^{\infty} a_i^{(1)} x_i\right\| - \left\|\sum_{i=p_1+1}^{\infty} a_i^{(1)} x_i\right\| \geq 1 - \frac{1}{2.4.K}.$$

2) Let take $y^{(k_2)}$ such that $\left\|\sum_{i=1}^{p_1} a_i^{(k_2)} x_i\right\| \leq \dfrac{1}{4.4^2.K}$. Find $p_2 \in \mathbb{N}$ such that $\left\|\sum_{i=p_2+1}^{\infty} a_i^{(k_2)} x_i\right\| \leq \dfrac{1}{4.4^2.K}$. Put $u_2 = \sum_{i=p_1+1}^{p_2} a_i^{(k_2)} x_i$ and $v_2 = u_2/\|u_2\|$. Obviously $\|v_2\| = 1$ and

$$1 \geq \|u_2\| = \left\|\sum_{i=1}^{\infty} a_i^{(k_2)} x_i - \sum_{i=1}^{p_1} a_i^{(k_2)} x_i - \sum_{i=p_2+1}^{\infty} a_i^{(k_2)} x_i\right\| \geq 1 - \frac{1}{4.4^2.K} - \frac{1}{4.4^2.K} = 1 - \frac{1}{2.4^2.K}.$$

3) Let take $y^{(k_3)}$ such that $\left\|\sum_{i=1}^{p_2} a_i^{(k_3)} x_i\right\| \leq \dfrac{1}{4.4^3.K}$. Find $p_3 \in \mathbb{N}$ such that $\left\|\sum_{i=p_3+1}^{\infty} a_i^{(k_3)} x_i\right\| \leq \dfrac{1}{4.4^3.K}$. Put $u_3 = \sum_{i=p_2+1}^{p_3} a_i^{(k_3)} x_i$ and $v_3 = u_3/\|u_3\|$. Obviously $\|v_3\| = 1$ and

$$1 \geq \|u_3\| = \left\|\sum_{i=1}^{\infty} a_i^{(k_3)} x_i - \sum_{i=1}^{p_2} a_i^{(k_3)} x_i - \sum_{i=p_3+1}^{\infty} a_i^{(k_3)} x_i\right\| \geq 1 - \frac{1}{4.4^3.K} - \frac{1}{4.4^3.K} = 1 - \frac{1}{2.4^3.K}.$$

If we have chosen p_{n-1}, u_{n-1} we proceed
4) Let take $y^{(k_n)}$ such that $\left\|\sum_{i=1}^{p_{n-1}} a_i^{(k_n)} x_i\right\| \leq \dfrac{1}{4.4^n.K}$. Find $p_n \in \mathbb{N}$ such that $\left\|\sum_{i=p_n+1}^{\infty} a_i^{(k_n)} x_i\right\| \leq \dfrac{1}{4.4^n.K}$. Put $u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i^{(k_n)} x_i$ and $v_n = u_n/\|u_n\|$. Obviously $\|v_n\| = 1$ and

$$1 \geq \|u_n\| = \left\|\sum_{i=1}^{\infty} a_i^{(k_n)} x_i - \sum_{i=1}^{p_{n-1}} a_i^{(k_n)} x_i - \sum_{i=p_n+1}^{\infty} a_i^{(k_n)} x_i\right\| \geq 1 - \frac{1}{4.4^n.K} - \frac{1}{4.4^n.K} = 1 - \frac{1}{2.4^n.K}.$$

Thus by

$$\begin{aligned}\sum_{n=1}^{\infty}\|y^{(k_n)}-v_n\| &= \sum_{n=1}^{\infty}\left\|y^{(k_n)}-u_n+u_n-\frac{u_n}{\|u_n\|}\right\| \leq \sum_{n=1}^{\infty}\|y^{(k_n)}-u_n\|+\sum_{n=1}^{\infty}\left\|u_n-\frac{u_n}{\|u_n\|}\right\| \\ &\leq \sum_{n=1}^{\infty}\left(\left\|\sum_{i=1}^{p_{n-1}}a_i^{(k_n)x_i}\right\|+\left\|\sum_{i=p_n+1}^{\infty}a_i^{(k_n)x_i}\right\|+\|u_n\|\left|1-\frac{1}{\|u_n\|}\right|\right) \\ &\leq \sum_{n=1}^{\infty}\left(\frac{1}{4.4^n.K}+\frac{1}{4.4^n.K}+\frac{1}{2.4^n.K}\right)=\sum_{n=1}^{\infty}\frac{1}{4^n.K}<\frac{1}{3.K}.\end{aligned}$$

follows that $\{y^{(k_n)}\}_{n=1}^{\infty}$ is equivalent to $\{v_n\}_{n=1}^{\infty}$. □

2 S_p and UT_q properties in weighted Orlicz sequence spaces

In this section we will investigate the S_p and UT_q–properties in weighted Orlicz sequence spaces.

It is found in [23] that all weighted Orlicz sequence spaces $\ell_M(w)$ are mutually isomorphic provided that $w \in \Lambda$. A sharp estimate the cotype of a weighted Orlicz sequence spaces $\ell_M(w)$ are found in [19], depending only on the Orlicz function M, provided that $w \in \Lambda$.

Naturally, the problem arises to find conditions for a weighted Orlicz sequence spaces $\ell_M(w)$ to have property S_p or UT_q, dependent only on the Orlicz function M, provided that $w \in \Lambda$.

There is no chance of mixing the Luxemburg norm in $\ell_M(w)$ and in $d(w, M)$, that is why in this section we will use $\|\cdot\|$ instead of $\|\cdot\|_{\ell_M(w)}$.

An equivalent definition of the S_p–property is the following:

Definition 2.1 *A X be a Banach space and $1 < p \leq \infty$. It is said that X has S_p–property if every weakly null normalized sequence $\{x_k\}_{k=1}^{\infty}$ has a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ so that for some constant $0 < C < \infty$ holds:*

$$\left\|\sum_{i=1}^{n}a_i x_{k_i}\right\| \leq C$$

for every $n \in \mathbb{N}$ and for all scalars $\{a_i\}_{i=1}^{n}$ with $\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p} \leq 1$. Here $(\sum|a_k|^{\infty})^{1/\infty} =$ $\max_{k\in\mathbb{N}}|a_k|$.

Indeed if the conditions in Definition 1.2 hold and $\{x_k\}_{k=1}^{\infty}$ is a weakly null normalized sequence, and $\{a_i\}_{i=1}^{\infty}$ be such that $(\sum_{i=1}^{n}|a_i|^p)^{1/p} \leq 1$, then $\left\|\sum_{i=1}^{n}a_i x_{k_i}\right\| \leq C\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p} \leq C$ for some subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ and some constant $C > 0$.

Let now hold the conditions in Definition 2.1 and $\{x_k\}_{k=1}^{\infty}$ be a weakly null normalized sequence, and $\{a_i\}_{i=1}^{\infty}$ be arbitrary sequence of scalars. Let define $\alpha_k = \dfrac{a_k}{(\sum_{i=1}^{n}|a_i|^p)^{1/p}}$. Then

$\sum_{k=1}^{n} |\alpha_k|^p = \sum_{k=1}^{n} \frac{|a_k|^p}{\sum_{i=1}^n |a_i|^p} = 1$. Thus $\|\sum_{i=1}^n \alpha_i x_{k_i}\| \leq C$ for some subsequence $\{x_{k_i}\}_{i=1}^\infty$ and some constant $C > 0$ or equivalently $\left\|\sum_{k=1}^{n} \frac{a_k}{\left(\sum_{i=1}^n |a_i|^p\right)^{1/p}} x_{k_i}\right\| = \left\|\sum_{i=1}^{n} \alpha_i x_{k_i}\right\| \leq C$. Thus we obtain $\|\sum_{i=1}^n a_i x_{k_i}\| \leq C \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}$.

Theorem 4 *Let $1 < p < \infty$, M is an Orlicz function, $w = \{w_i\}_{i=1}^\infty \in \Lambda$. Let $h_M(w)$ be a weighted Orlicz sequence space not containing ℓ_1. Then the following are equivalent:*
(a) $h_M(w)$ has property BS_p;
(b) the Orlicz function M satisfies:

$$\sup_{0<s,t\leq 1} \frac{M(st)}{M(s)t^p} < \infty;$$

(c) $h_M(w)$ has property S_p.

Proof: (c)⇒(a) follows by Proposition 1.2.

(a)⇒(b) WLOG we may assume that

$$\lim_{j\to\infty} w_j = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} w_j = \infty.$$

If not we can consider the subspace $h_M(\{w_{j_k}\}_{k=1}^\infty) \hookrightarrow h_M(w)$.

For any $k \in \mathbb{N}$ there are sequences $\{p_m^{(k)}\}_{m=1}^\infty$ and $\{q_m^{(k)}\}_{m=1}^\infty$ such that

$$1 \leq p_1^{(k)} \leq q_1^{(k)} < p_2^{(k)} \leq q_2^{(k)} < \ldots < p_m^{(k)} \leq q_m^{(k)} < \ldots$$

and

$$k - \frac{1}{2} < \sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j < k.$$

Let the sequence $\{\alpha_k\}_{k=1}^\infty$ be the solution of the equation $kM(\alpha_k) = 1$. Obviously $\alpha_k \searrow 0$.

Let $b_m^{(k)} = \sum_{j=p_m^{(k)}}^{q_m^{(k)}} \alpha_k e_j$. The sequence $\{b_m^{(k)}\}_{m=1}^\infty$ is a weakly null sequence for every $k \in \mathbb{N}$ and $\|b_m^{(k)}\| \leq 1$.

Let define now $y_m^{(k)} = \frac{b_m^{(k)}}{\|b_m^{(k)}\|}$. Obviously $\|y_m^{(k)}\| = 1$. Thus there exists an increasing sequence of naturals $\{m_s\}_{s=1}^\infty$ such that:

$$\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \leq \left\|\sum_{s=1}^{n} \frac{b_{m_s}^{(k)}}{\|b_{m_s}^{(k)}\|}\right\| = \left\|\sum_{s=1}^{n} y_{m_s}^{(k)}\right\| \leq C_1 n^{1/p}$$

holds for some constant $C_1 < \infty$ and for every $n \in \mathbb{N}$.

By the inequalities:

$$\begin{aligned}\sum_{m=1}^{n}\sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j \; M\left(\frac{\alpha_k}{2\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) &\leq \sum_{m=1}^{n}\frac{1}{2}M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right)\sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j \leq \sum_{m=1}^{n}\frac{k}{2}M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right)\\ &= \frac{nk}{2}M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \leq n(k-1/2)M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right)\\ &= \sum_{m=1}^{n}(k-1/2)M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \leq \sum_{s=1}^{n}\sum_{j=p_{m_s}^{(k)}}^{q_{m_s}^{(k)}} w_j M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \leq 1\end{aligned}$$

follows that $\left\|\sum_{m=1}^{n} b_m^{(k)}\right\| \leq 2\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \leq Cn^{1/p}$, where $C = 2C_1$.

The last inequality is equivalent to

$$1 \geq \sum_{m=1}^{n}\sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j M\left(\frac{\alpha_k}{Cn^{1/p}}\right) \geq n\left(k-\frac{1}{2}\right)M\left(\frac{\alpha_k}{Cn^{1/p}}\right). \tag{2}$$

WLOG we may assume that $s < \alpha_1$ and $t < 1/C$. Let $s, t \in (0,1]$, there exists $k \in \mathbb{N}$ such that $\alpha_{k+1} < s \leq \alpha_k$ and there exists $n \in \mathbb{N}$ such that $\frac{1}{C(n+1)^{1/p}} < t \leq \frac{1}{Cn^{1/p}}$. So

$$\frac{M(st)}{M(s)t^p} \leq \frac{M\left(\frac{\alpha_k}{Cn^{1/p}}\right)}{M(\alpha_{k+1})\left(\frac{1}{C(n+1)^{1/p}}\right)^p} \leq \frac{(k+1)(n+1)C^p}{n\left(k-\frac{1}{2}\right)} \leq 8C^p$$

(b)⇒(c) Let $\{y^{(k)}\}_{k=1}^{\infty}$ be a sequence defined by $y^{(k)} = \sum_{j=p_k}^{q_k} y_j e_j$, where $1 \leq p_1 \leq q_1 < p_2 \leq q_2 < \ldots < p_k \leq q_k < \ldots$ and $\sum_{j=p_k}^{q_k} w_j M(y_j) \leq 1$. By (b) there is a constant $0 < C < \infty$ such that

$$\sup_{0<s,t\leq 1}\frac{M(st)}{M(s)t^p} \leq C^p.$$

Let $\{\beta_k\}_{k=1}^{\infty}$ be such that $\sum_{k=1}^{\infty}|\beta_k|^p \leq 1$. We will show that $\left\|\sum_{i=1}^{\infty}\beta_k y^{(k)}\right\| \leq C$. Indeed:

$$\sum_{k=1}^{\infty}\sum_{j=p_k}^{q_k} w_j M\left(\frac{\beta_k y_j}{C}\right) \leq \sum_{k=1}^{\infty}\sum_{j=p_k}^{q_k} w_j|\beta_k|^p M(y_j) \leq \sum_{k=1}^{\infty}|\beta_k|^p\sum_{j=p_k}^{q_k} w_j M(y_j) \leq \sum_{k=1}^{\infty}|\beta_k|^p \leq 1.$$

Since every weakly null normalized sequence $\{x_i\}_{i=1}^{\infty}$ in $h_M(w)$ has a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ which is equivalent to a normalized block basic sequence $\{y^{(k)}\}_{k=1}^{\infty}$, the proof is finished. $\square$

Theorem 5 *Let $1 < q < \infty$, M is an Orlicz function, $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Let $h_M(w)$ be a weighted Orlicz sequence space not containing ℓ_1. Then the following are equivalent:*
(a) $h_M(w)$ has property UT_q;
(b) the Orlicz function M satisfies:

$$\inf_{0<s,t\leq 1} \frac{M(st)}{M(s)t^p} > 0.$$

Proof: (a)$\Rightarrow$(b) Consider $C_1 > 0$ such that every weakly null normalized sequence admits a subsequence which has a lower q–estimate with a constant C_1. As mentioned in the proof of Theorem 4 WLOG we may assume that

$$\lim_{j\to\infty} w_j = 0 \text{ and } \sum_{j=1}^{\infty} w_j = \infty.$$

For any $k \in \mathbb{N}$ there are sequences $\{p_m^{(k)}\}_{m=1}^{\infty}$ and $\{q_m^{(k)}\}_{m=1}^{\infty}$ such that

$$1 \leq p_1^{(k)} \leq q_1^{(k)} < p_2^{(k)} \leq q_2^{(k)} < \ldots < p_m^{(k)} \leq q_m^{(k)} < \ldots$$

and

$$k - \frac{1}{2} < \sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j < k.$$

Let the sequence $\{\alpha_k\}_{k=1}^{\infty}$ be the solution of the equation $kM(\alpha_k) = 1$. Obviously $\alpha_k \searrow 0$.

Let $b_m^{(k)} = \sum_{j=p_m^{(k)}}^{q_m^{(k)}} \alpha_k e_j$. The sequence $\{b_m^{(k)}\}_{m=1}^{\infty}$ is a weakly null sequence for every $k \in \mathbb{N}$ and obviously $\|b_m^{(k)}\| \leq 1$. By the inequalities:

$$\sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j M(2\alpha_k) \geq 2M(\alpha_k) \sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j \geq 2\left(k - \frac{1}{2}\right) M(\alpha_k) \geq kM(\alpha_k) = 1$$

follows that $\|b_m^{(k)}\| \geq 1/2$. Let define now $y_m^{(k)} = \dfrac{b_m^{(k)}}{\|b_m^{(k)}\|}$. Obviously $\|y_m^{(k)}\| = 1$. Thus there exists an increasing sequence of naturals $\{m_s\}_{s=1}^{\infty}$ such that:

$$2\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \geq \left\|\sum_{s=1}^{n} \frac{b_{m_s}^{(k)}}{\|b_{m_s}^{(k)}\|}\right\| = \left\|\sum_{s=1}^{n} y_{m_s}^{(k)}\right\| \geq C_1 \left(\sum_{s=1}^{n} 1^q\right)^{1/q} = C_1 n^{1/q}$$

holds for some constant $C_1 < \infty$ and for every $n \in \mathbb{N}$. Then

$$\left\| \sum_{s=1}^{n} b_{m_s}^{(k)} \right\| \geq C_2 n^{1/q}$$

for every $n \in \mathbb{N}$ with $C_2 = C_1/2$.

By the inequalities:

$$\begin{aligned}
\sum_{m=1}^{n} \sum_{j=p_m^{(k)}}^{q_m^{(k)}} \quad w_j \quad M\left(\frac{2\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) &\geq \sum_{m=1}^{n} 2M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j \\
&\geq \quad \sum_{m=1}^{n} 2\left(k-\frac{1}{2}\right) M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) = 2n\left(k-\frac{1}{2}\right) M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \\
&\geq \quad nkM\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) = \sum_{m=1}^{n} kM\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) \geq \sum_{s=1}^{n} \sum_{j=p_{m_s}^{(k)}}^{q_{m_s}^{(k)}} w_j M\left(\frac{\alpha_k}{\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\|}\right) = 1
\end{aligned}$$

follows that $\left\| \sum_{m=1}^{n} b_m^{(k)} \right\| \geq \frac{1}{2} \left\| \sum_{s=1}^{n} b_{m_s}^{(k)} \right\| \geq C n^{1/q}$, where $C = \dfrac{C_2}{2}$.

The last inequality is equivalent to

$$1 \leq \sum_{m=1}^{n} \sum_{j=p_m^{(k)}}^{q_m^{(k)}} w_j M\left(\frac{\alpha_k}{Cn^{1/q}}\right) \leq nkM\left(\frac{\alpha_k}{Cn^{1/q}}\right). \tag{3}$$

WLOG we may assume that $s < \alpha_1$ and $t < 1/C$ Now for every $s, t \in (0, 1]$. There exists $k \in \mathbb{N}$ such that $\alpha_{k+1} < s \leq \alpha_k$ and there exists $n \in \mathbb{N}$ such that $\dfrac{1}{C(n+1)^{1/q}} < t \leq \dfrac{1}{Cn^{1/q}}$. So

$$\frac{M(st)}{M(s)t^q} \geq \frac{M\left(\dfrac{\alpha_{k+1}}{(C(1+n)^{1/q})^q}\right)}{M(\alpha_k)\left(\dfrac{1}{Cn^{1/q}}\right)^q} \geq \frac{knC^q}{(k+1)(n+1)} \geq \frac{1}{4}C^q$$

(b)⇒(a) Let hold (b) and let $y^{(k)} = \sum_{i=p_k}^{q_k} \alpha_i e_i$ be a sequence with $1 \leq p_1 \leq q_1 < p_2 \leq q_2 < \ldots < p_k \leq q_k < p_{k+1} \leq \ldots$. Let

$$\inf_{0<s,t\leq 1} \frac{M(st)}{M(s)t^q} > C_1,$$

then

$$\sum_{k=1}^{n}\sum_{i=p_k}^{q_k} \quad w_i \quad M\left(\frac{\alpha_i}{\|y^{(k)}\|}\cdot\frac{\|y^{(k)}\|}{C_1^{1/q}\left(\sum_{j=1}^{n}\|y^{(j)}\|^q\right)^{1/q}}\right)\geq\sum_{k=1}^{n}\sum_{i=p_k}^{q_k} w_i M\left(\frac{\alpha_i}{\|y^{(k)}\|}\right)\cdot\frac{\|y^{(k)}\|^q}{\sum_{j=1}^{n}\|y^{(j)}\|^q}$$

$$= \quad \frac{1}{\sum_{j=1}^{n}\|y^{(j)}\|^q}\sum_{k=1}^{n}\|y^{(k)}\|^q\sum_{i=p_k}^{q_k} w_i M\left(\frac{\alpha_i}{\|y^{(k)}\|}\right)=1.$$

Thus we have the inequality $C_1\left(\sum_{i=1}^{n}\|y^{(k)}\|^q\right)^{1/q}\leq\left\|\sum_{k=1}^{n}y^{(k)}\right\|$

Since every weakly null normalized sequence $\{x_i\}_{i=1}^{\infty}$ in $h_M(w)$ has a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ which is equivalent to a normalized block basic sequence $\{y^{(k)}\}_{k=1}^{\infty}$ we get for any sequence $\{\alpha_k\}_{k=1}^{\infty}$ of scalars

$$\left\|\sum_{k=1}^{n}\alpha_k x_{i_k}\right\|\geq C_2\left\|\sum_{k=1}^{n}\alpha_k y^{(k)}\right\|\geq C_1C_2\left(\sum_{k=1}^{n}\|\alpha_k y^{(k)}\|^q\right)^{1/q}=C_1C_2\left(\sum_{k=1}^{n}|\alpha_k|^q\right)^{1/q}$$

and thus $\{x_{i_k}\}_{i=1}^{\infty}$ has a lower q–estimate with a constant $C=C_1C_2$. □

Related to these properties are the following indexes defined in [10]:

$$\ell(X)=\sup\{p\geq 1: X \text{ has } S_p - \text{ property}\}$$

$$u(X)=\inf\{q\geq 1: X \text{ has } T_q - \text{ property}\}.$$

Obviously $\ell(\ell_p)=u(\ell_p)=p$. It is found in [8] that $\ell(h_M)=\alpha_M$, $u(h_M)=\beta_M$ and $\ell(d_0(w,p))=p$, where α_M and β_M are the Boyd indexes associated to M

$$\alpha_M=\sup\left\{p\geq 1:\sup_{0<u,v\leq 1}\frac{M(uv)}{u^pM(v)}<\infty\right\}$$

$$\beta_M=\inf\left\{q\geq 1:\inf_{0<u,v\leq 1}\frac{M(uv)}{u^qM(v)}>0\right\}.$$

Corollary 2.1 *Let $h_M(w)$ be a weighted Orlicz sequence space generated by an Orlicz function M and a weight sequence $w=\{w_i\}_{i=1}^{\infty}\in\Lambda$. Then $\ell(h_M(w))=\alpha_M$ and $u(h_M(w))=\beta_M$.*

Proof: From Theorem 4 and Theorem 5 and since $h_M(w)$ contains an isomorphic copy of ℓ_{α_M} and ℓ_{β_M}, the result follows. □

The exact upper and lower estimates are found for the unit vector basis $\{e_i\}_{i=1}^{\infty}$ in h_M in terms of the generating function M. When dealing with a weight sequence the problem is much more complicated. Some upper estimates are done for the unit vector basis in Lorentz sequence space $d_0(w,p)$ in terms of the weight sequence $w=\{w_i\}_{i=1}^{\infty}$.

Next by using the ideas from [8] we will find some upper estimates of the unit vector basis $\{e_i\}_{i=1}^{\infty}$ in h_M in terms of the generating Orlicz function M and the sequence $w=\{w_i\}_{i=1}^{\infty}$.

Let start with the following notation:

For $1 < r < \infty$ the number r^* is the solution of the equation $\dfrac{1}{r} + \dfrac{1}{r^*} = 1$. We will use the standard notation $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$ for the norm in ℓ_p.

Proposition 2.1 *Let $h_M(w)$ be a weighted Orlicz sequence space generated by an Orlicz function M and a weight sequence $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then holds:*
(a) If $\sup\limits_{0<t\leq 1} \dfrac{M(t)}{t^p} < \infty$ and $w \in \ell_{(s/p)^*}$ for some $s > p$ then the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate;
(b) If $\inf\limits_{0<t\leq 1} \dfrac{M(t)}{t^q} > 0$ and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate then for some $s > q$ then $w \in \ell_{(s/q)^*}$.

Proof: Let $w \in \ell_{(s/p)^*}$ for some $s > p$.

There exist a constant $C > 0$ such that $\dfrac{M(t)}{t^p} < C$ for every $t \in (0, 1]$.

Consider the sequence $a = \{a_i\}_{i=1}^{\infty} \in \ell_s$ such that $\|a\|_s \leq 1$. Then for each $n \in \mathbb{N}$ we have

$$1 = \sum_{i=1}^{n} w_i M\left(\frac{a_i}{\|\sum_{i=1}^{n} a_i e_i\|_{\ell_M(w)}}\right) \leq C \sum_{i=1}^{n} w_i \frac{|a_i|^p}{\|\sum_{i=1}^{n} a_i e_i\|_{\ell_M(w)}^p} \tag{4}$$

and thus follows

$$\left\|\sum_{i=1}^{n} a_i e_i\right\|_{\ell_M(w)}^p \leq C \sum_{i=1}^{n} w_i |a_i|^p \leq C \left(\sum_{i=1}^{n} w_i^{(s/p)^*}\right)^{(p/s)^*} \left(\sum_{i=1}^{n} |a_i|^s\right)^{(p/s)}.$$

Hence $\|\sum_{i=1}^{n} a_i e_i\|_{\ell_M(w)} \leq C \|w\|_{(s/p)^*}^{1/p} \|a\|_s$.

(b) If the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate then $\|\sum_{i=1}^{n} a_i e_i\|_{\ell_M(w)} \leq L \left(\sum_{i=1}^{\infty} |a_i|^s\right)^{1/s}$ holds for some constant L and for any $a_1, a_2, \ldots, a_n$, $n \in \mathbb{N}$.

There exist a constant $c > 0$ such that $\dfrac{M(t)}{t^q} > c$ for every $t \in (0, 1]$.

Let $b = \{b_i\}_{i=1}^{\infty} \in \ell_{s/q}$. By the inequalities

$$\sum_{i=1}^{n} w_i M\left(\frac{|b_i|^{1/q}}{c^{1/q} \left(\sum_{i=1}^{n} w_i |b_i|\right)^{1/q}}\right) \geq \sum_{i=1}^{n} w_i \frac{|b_i|}{\sum_{i=1}^{n} w_i |b_i|} = 1$$

and the definition of the Luxemburg norm in $h_M(w)$ follows that

$$c^{1/q} \left(\sum_{i=1}^{n} w_i |b_i|\right)^{1/q} \leq \left\|\sum_{i=1}^{n} |b_i|^{1/q} e_i\right\|_{\ell_M(w)} \leq L \left(\sum_{i=1}^{\infty} |b_i|^{s/q}\right)^{1/s} \leq L \|b\|_{s/q}^{1/q}.$$

Hence $\sum_{i=1}^{\infty} |b_i| w_i \leq \dfrac{L^q}{c} \|b\|_{s/q}$ for any $b \in \ell_{s/q}$ and therefore $w \in \ell_{(s/q)^*}$. □

Remark: Let us mention that if $\sup\limits_{0<t\leq 1}\dfrac{M(t)}{t^p}<\infty$ then the unit vector basis $\{e_n\}_{n=1}^\infty$ has an upper p–estimate. Indeed $w\in\ell_\infty$ and thus s should be equal to p. The proof follows right away taking into account in (4) that $\sup_{i\in\mathbb{N}} w_i<\infty$. As stated in the theorem there are cases when the unit vector basis in $h_M(w)$ may have a better upper s–estimate for $s>p$, depending on the weight sequence $w=\{w_i\}_{i=1}^\infty$.

If $M(t)=t^p$, the result is obtained in [8].

3 S_p property in Lorenz–Orlicz sequence spaces

Recently a deep result on embedding of ℓ_p spaces in Lorentz–Orlicz sequence spaces have been found in [13]. It is shown there that $\ell_p\hookrightarrow d_0(w,M)$ iff $\ell_p\hookrightarrow h_M$ iff $p\in[\alpha_M,\beta_M]$. This result naturally arises the question for finding of upper and lower p–estimates in Lorentz–Orlicz sequence spaces.

There is no chance of mixing the Luxemburg norm in $d(w,M)$ and in $\ell_M(w)$, that is why in this section we will use $\|\cdot\|$ instead of $\|\cdot\|_{d(w,M)}$.

Theorem 6 *Let $d_0(w,M)$ be a Lorentz–Orlicz sequence space generated by an Orlicz function M and a weight sequence $w=\{w_i\}_{i=1}^\infty$ satisfying $w_i\searrow 0$ and $\sum_{i=1}^\infty w_i=\infty$, not having an isomorphic copy of ℓ_1. Then:*

(a) If $\sup\limits_{0<s,t\leq 1}\dfrac{M(st)}{M(s)t^p}<\infty$ for some $1<p<\infty$ then $d_0(w,M)$ has property S_p;

(b) If $d_0(w,M)$ has property UT_q for some $1<q<\infty$ then $\inf\limits_{0<s,t\leq 1}\dfrac{M(st)}{M(s)t^q}>0$.

Proof: (a) Let $\{x_i\}_{i=1}^\infty$ be a weakly null normalized sequence in $d_0(w,M)$. It has a subsequence $\{x_{i_k}\}_{k=1}^\infty$ which is equivalent to a normalized block basic sequence $y^{(k)}=\sum_{j=p_k+1}^{p_{k+1}} y_j e_j$, i.e. $\sum\limits_{j=p_k+1}^{p_{k+1}} w_j M(y_j)=1$.

There is a constant $0<C<\infty$ such that

$$\sup_{0<s,t\leq 1}\frac{M(st)}{M(s)t^p}\leq C^p.$$

Let $\{\beta_k\}_{k=1}^\infty$ be such that $\sum_{k=1}^\infty|\beta_k|^p\leq 1$. We will show that $\left\|\sum_{i=1}^\infty \beta_k y^{(k)}\right\|\leq C$. Indeed:

$$\begin{aligned}\sum_{k=1}^\infty\sum_{j=p_k+1}^{p_{k+1}} w_j M\left(\frac{\beta_k y_j}{C}\right) &\leq \sum_{k=1}^\infty\sum_{j=1}^{p_{k+1}-p_k} w_j M\left(\frac{\beta_k y_{p_k+j}}{C}\right)\\ &\leq \sum_{k=1}^\infty\sum_{j=1}^{p_{k+1}-p_k} w_j|\beta_k|^p M(y_{p_k+j})\\ &\leq \sum_{k=1}^\infty|\beta_k|^p\sum_{j=1}^{p_{k+1}-p_k} w_j M(y_{p_k+j})\leq 1.\end{aligned}$$

(b) Consider $C_1 > 0$ such that every weakly null normalized sequence admits a subsequence which has a lower q–estimate with a constant C_1.

For any $k \in \mathbb{N}$ there exist p_k such that

$$k - \frac{1}{2} < \sum_{j=1}^{p_k} w_j < k.$$

Let the sequence $\{\alpha_k\}_{k=1}^{\infty}$ be the solution of the equation $kM(\alpha_k) = 1$. Obviously $\alpha_k \searrow 0$.

Let $b_m^{(k)} = \sum_{j=1+(m-1)p_k}^{mp_k} \alpha_k e_j$ for $m \in \mathbb{N}$. The sequence $\{b_m^{(k)}\}_{m=1}^{\infty}$ is a weakly null sequence for every $k \in \mathbb{N}$ and $\|b_m^{(k)}\| \leq 1$. By the inequalities:

$$\sum_{j=1}^{p_k} w_j M(2\alpha_k) \geq 2M(\alpha_k) \sum_{j=1}^{p_k} w_j \geq 2\left(k - \frac{1}{2}\right) M(\alpha_k) \geq kM(\alpha_k) = 1$$

follows that $\|b_m^{(k)}\| \geq 1/2$. Let define now $y_m^{(k)} = \dfrac{b_m^{(k)}}{\|b_m^{(k)}\|}$. Obviously $\|y_m^{(k)}\| = 1$. Thus there exists an increasing sequence of naturals $\{m_s\}_{s=1}^{\infty}$ such that:

$$2\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \geq \left\|\sum_{s=1}^{n} \frac{b_{m_s}^{(k)}}{\|b_{m_s}^{(k)}\|}\right\| = \left\|\sum_{s=1}^{n} y_{m_s}^{(k)}\right\| \geq C_1 \left(\sum_{s=1}^{n} 1^q\right)^{1/q} = C_1 n^{1/q}$$

holds for every $n \in \mathbb{N}$. Then

$$\left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \geq Cn^{1/q}$$

for every $n \in \mathbb{N}$ with $C = C_1/2$.

By the symmetry of the unit vector basis follows that $\left\|\sum_{m=1}^{n} b_m^{(k)}\right\| = \left\|\sum_{s=1}^{n} b_{m_s}^{(k)}\right\| \geq Cn^{1/q}$.

The last inequality is equivalent to

$$1 \leq \sum_{m=1}^{n} \sum_{j=1+(m-1)p_k}^{mp_k} w_j M\left(\frac{\alpha_k}{Cn^{1/q}}\right) \leq \sum_{m=1}^{n} \sum_{j=1}^{p_k} w_j M\left(\frac{\alpha_k}{Cn^{1/q}}\right) \leq nkM\left(\frac{\alpha_k}{Cn^{1/q}}\right). \tag{5}$$

WLOG we may assume that $s < \alpha_1$ and $t < 1/C$. Now for every $s, t \in (0, 1]$, there exists $k \in \mathbb{N}$ such that $\alpha_{k+1} < s \leq \alpha_k$ and there exists $n \in \mathbb{N}$ such that $\dfrac{1}{C(n+1)^{1/q}} < t \leq \dfrac{1}{Cn^{1/q}}$. So

$$\frac{M(st)}{M(s)t^q} \geq \frac{M\left(\dfrac{\alpha_{k+1}}{(C(1+n)^{1/q})^q}\right)}{M(\alpha_k)\left(\dfrac{1}{Cn^{1/q}}\right)^q} \geq \frac{knC^q}{(k+1)(n+1)} \geq \frac{1}{4}C^q$$

□

Remark: If $M(t) = t^p$ the result is obtained in [3].

Corollary 3.1 *Let $d_0(w,M)$ be a Lorentz–Orlicz sequence space generated by an Orlicz function M and a weight sequence $w = \{w_i\}_{i=1}^{\infty}$ satisfying $w_i \searrow 0$ and $\sum_{i=1}^{\infty} w_i = \infty$. Then $\ell(d_0(w,M)) = \alpha_M$.*

Proof: From Theorem 6 and since $d_0(w,M)$ contains an isomorphic copy of ℓ_{α_M}, the result follows. □

Proposition 3.1 *Let $d_0(w,M)$ be a Lorentz–Orlicz sequence space generated by an Orlicz function M and a weight sequence $w = \{w_i\}_{i=1}^{\infty}$ satisfying $w_i \searrow 0$ and $\sum_{i=1}^{\infty} w_i = \infty$. Then*
(a) If $\sup\limits_{0<t\leq 1} \dfrac{M(t)}{t^p} < \infty$ and $w \in \ell_{(s/p)^*}$ for some $s > p$ then the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate;
(b) If $\inf\limits_{0<t\leq 1} \dfrac{M(t)}{t^q} > 0$ and the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate then for some $s > q$ then $w \in \ell_{(s/q)^*}$.

Proof: Let $w \in \ell_{(s/p)^*}$ for some $s > p$.

There exist a constant $C > 0$ such that $\dfrac{M(t)}{t^q} < C$ for every $t \in (0,1]$.

Consider the sequence $a = \{a_i\}_{i=1}^{\infty} \in \ell_s$ such that $\|a\|_s \leq 1$. Then for each $n \in \mathbb{N}$ we have

$$1 = \sum_{i=1}^{n} w_i M\left(\frac{a_i^*}{\|\sum_{i=1}^{n} a_i e_i\|_{d_0(w,M)}}\right) \leq C \sum_{i=1}^{n} w_i \frac{|a_i^*|^p}{\|\sum_{i=1}^{n} a_i e_i\|_{d_0(w,M)}^p}$$

and thus follows

$$\left\|\sum_{i=1}^{n} a_i e_i\right\|_{d_0(w,M)}^p \leq C \sum_{i=1}^{n} w_i |a_i^*|^p \leq C \left(\sum_{i=1}^{n} w_i^{(s/p)^*}\right)^{(p/s)^*} \left(\sum_{i=1}^{n} |a_i^*|^s\right)^{(p/s)}.$$

Hence $\|\sum_{i=1}^{n} a_i e_i\|_{d_0(w,M)} \leq C \|w\|_{(s/p)^*}^{1/p} \|a\|_s$.

(b) If the unit vector basis $\{e_n\}_{n=1}^{\infty}$ has an upper s–estimate then $\|\sum_{i=1}^{n} a_i e_i\| \leq L \left(\sum_{i=1}^{\infty} |a_i|^s\right)^{1/s}$ holds for some constant L and for any $a_1, a_2, \ldots, a_n$, $n \in \mathbb{N}$.

There exist a constant $c > 0$ such that $\dfrac{M(t)}{t^q} > c$ for every $t \in (0,1]$.

Let $b = \{b_i\}_{i=1}^{\infty} \in \ell_{s/q}$. By the inequalities

$$\sum_{i=1}^{n} w_i M\left(\frac{(b_i^*)^{1/q}}{c^{1/q} \left(\sum_{i=1}^{n} w_i b_i^*\right)^{1/q}}\right) \geq \sum_{i=1}^{n} w_i \frac{b_i^*}{\sum_{i=1}^{n} w_i b_i^*} = 1$$

and the definition of the Luxemburg norm in $d_0(w,M)$ follows that

$$c^{1/q} \left(\sum_{i=1}^{n} w_i |b_i|\right)^{1/q} \leq c^{1/q} \left(\sum_{i=1}^{n} w_i b_i^*\right)^{1/q} \leq \left\|\sum_{i=1}^{n} |b_i|^{1/q} e_i\right\|_{d_0(w,M)} \leq L \left(\sum_{i=1}^{\infty} |b_i|^{s/q}\right)^{1/s} \leq L \|b\|_{s/q}^{1/q}.$$

Hence $\sum_{i=1}^{\infty} |b_i| w_i \leq \dfrac{L^q}{c} \|b\|_{s/q}$ for any $b \in \ell_{s/q}$ and therefore $w \in \ell_{(s/q)^*}$. □

Remark: If $M(t) = t^p$ the result is obtained in [8].

4 Some applications to polynomials

As mentioned in the introduction upper and lower p–estimates give us much information about weak continuity of polynomials and this is of great interest in some problems of smoothness [10], [4], [11], [18]. A connection between upper and lower estimates in a Banach space and weak continuity of polynomials is obtained in [8]:

Proposition 4.1 *[8] Let X be a Banach space.*
(i) If a sequence $\{x_n\}_{n=1}^{\infty}$ in X has an upper p–estimate, then for every N–homogeneous polynomial P on X, with $N < p$, the sequence $\{P(x_n)\}_{n=1}^{\infty}$ is convergent to zero.
(ii) If X has a basis $\{e_n\}_{n=1}^{\infty}$ which satisfies a lower q–estimate, then there exists an N–homogeneous polynomial on X, with $N \geq q$, such that $P(e_n) \geq 1$ for all $n \in \mathbb{N}$.

By Proposition 4.1 and the results of the previous sections follows:

Corollary 4.1 *Let $h_M(w)$ be a weighted Orlicz sequence space generated by an Orlicz function M and a weight sequence $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ and not containing an isomorphic copy of ℓ_1. Then any N–homogeneous polynomial such that $N < \alpha_M$ is weakly sequentially continuous. Moreover if $N < s$, where $s > p$, $\sup\limits_{0<t\leq 1} \dfrac{M(t)}{t^p} < \infty$ and $w \in \ell_{(s/p)^*}$ then for every N–homogeneous polynomial P the sequence $\{P(e_n)\}_{n=1}^{\infty}$ converges to zero.*

Corollary 4.2 *Let $d_0(w, M)$ be a Lorentz–Orlicz sequence space generated by an Orlicz function M and a weight sequence $w = \{w_i\}_{i=1}^{\infty}$ satisfying $w_i \searrow 0$ and $\sum_{i=1}^{\infty} w_i = \infty$ and not containing an isomorphic copy of ℓ_1. Then any N–homogeneous polynomial such that $N < \alpha_M$ is weakly sequentially continuous.Then for every N–homogeneous polynomial P the sequence $\{P(e_n)\}_{n=1}^{\infty}$ converges to zero, provided $N < s$, where $s > p$, $\sup\limits_{0<t\leq 1} \dfrac{M(t)}{t^p} < \infty$ and $w \in \ell_{(s/p)^*}$.*

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