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ESTIMATES OF THE CORRECTION COEFFICIENT IN COULOMB'S LAW FOR ELECTROSTATIC INTERACTION BETWEEN TWO CHARGED CONDUCTING SPHERES

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Abstract

In the present work we consider the coefficient (correction coefficient), which compliments Coulomb's Law in the case of electrostatic interaction between two charged conducting spheres with equal radii and charges. It is proved that the correction coefficient is smaller than one, when the ratio of the radii to the distance between their centers is smaller than 2/5. We obtained a formula for calculating the correction coefficient with an arbitrary precision with the help of partial sums.

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1 Introduction

The Coulomb's Law defines the force of electrostatic interaction between two point charges. In practice, however, the interaction is not between point charges but between charged bodies, which have specific dimensions, geometry and physical structure. This causes the problem for finding the actual electrostatic force between two charged bodies.

The problem of determining the electrostatic force of interaction between two charged conducting spheres with arbitrary radiuses and charges was first investigated by Poisson. Later, Sir Thompson (Lord Kelvin) introduces his image charges theory thus significantly simplifying the investigation. This problem is later on considered by Maxwell ([7], Chapter 1). He discovers that the electrostatic force between the two spheres is different from the electrostatic force between the point charges (with the same magnitude and sign) located at the centres of the spheres,

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which is derived by Coulomb's law. According to Maxwell this deviation is caused by the redistribution of the charges as a result of the mutual electrostatic influence between the spheres. Having in mind the redistribution of the charges, Maxwell suggests a general method for determining the force of interaction between two spheres with arbitrary charges and radii using zonal harmonics ([7], Chapters 11-13). Many scientists after him find other solutions to the problem. In different special cases they derive exact formulas or give approximate formulas which are good enough for the solution of some theoretical or practical problems. Soules analyses Coulomb's trials and conducts precise experiments having in mind the induction effects [12]. He uses the image charges method to develop a computer program in order to determine numerically the force of interaction. Based on the analysis of the numerical values Soules also suggests an approximated formula for the electrostatic force. A number of authors using the image charges method derive approximate formulas for the force of interaction between two charged conducting spheres in the special case when they are with equal radiuses and charges. Such a formula is found by Slisko and Brito-Orta and using a computer program they compare the values calculated using different approximations [13].

Recent results on the electrostatic force of interaction between two charged conducting bodies are obtained in [1], [2], [3], [5], [8], [9], [10], [11].

An exact analytical formula for the force of electrostatic interaction between two spheres with arbitrary radii and charges is obtained in [4]. There a correction coefficient is introduced, which complements Coulomb's law, represented by the double infinite sum. In the present work we prove the convergence of the correction coefficient for the particular case of two conducting spheres with equal radii and equal charges. At predetermined correction coefficient error, we determine the upper limits of summation indexes in the calculation of this coefficient, when replacing the infinite sum with finite one.

We will consider in the present work two conductive spheres with equal radii r and equal charges Q located at a distance $R > 2r$ between their centers. Let us denote $\delta = \frac{r}{R}$. We will consider the following functions

$$f(\delta) = \sqrt{1-4\delta^2}, \quad C_j(\delta) = \frac{(1+f(\delta))^{j+1} - (1-f(\delta))^{j+1}}{2^{j+1} f(\delta)},$$

$$\phi_j(\delta) = \frac{\delta^2 C_{j-1}(\delta)}{C_j(\delta)}, \quad \Phi_{i,j}(\delta) = \frac{1}{(1-\phi_i(\delta)-\phi_j(\delta))^2},$$

$$L_0(\delta) = \frac{1}{1 + \sum\limits_{m=1}^{\infty} \frac{\delta^{2m}}{C_{2m}(\delta)} - \sum\limits_{m=1}^{\infty} \frac{\delta^{2m-1}}{C_{2m-1}(\delta)}}, \quad L_j(\delta) = \frac{(-\delta)^j}{C_j(\delta)} L_0(\delta)$$

and

$$L(\delta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} L_i(\delta) L_j(\delta) \Phi_{i,j}(\delta), \tag{1}$$

that are defined for $\delta \in \left[0, \frac{1}{2}\right)$ and the index $j \in \mathbb{N} \cup \{0\}$.

We have obtained in [4] the formula

$$F = \frac{Q^2}{4\pi\varepsilon_0 R^2} L(\delta), \tag{2}$$

by applying the image charges method to the electric field created by the electric charges of the two conducting spheres, which presents the force of electrostatic interaction between the two spheres. It is easy to see that if we consider two point charges, i.e. $\delta = 0$ in (1), then $L(0) = 1$, which is the Coulomb's Law.

We will use the Lambert W function, which presents the solution of the equation $y = W(y)e^{W(y)}$. The Lambert W function is defined for any complex number y. We will restrict to the real valued branch of $W : \left[-\frac{1}{e}, \infty\right) \to \mathbb{R}$.

Let $A = \left(a_i^j\right)$, $i, j = 0, 1, 2, \ldots$ be an infinite matrix of real numbers. Let $\{u_k\}_{k=0}^{\infty}$ be an arbitrary representation the matrix A as a sequence.

Theorem 1. *([6], p. 362) If the double series* $\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left|a_i^j\right|$ *is convergent then the series* $\sum_{k=0}^{\infty} u_k$ *is convergent for any representation* $\{u_k\}_{k=0}^{\infty}$ *of the matrix* A *and*

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i^j = \sum_{k=0}^{\infty} u_k. \tag{3}$$

We will use a diagonal representation $\sum_{j=0}^{\infty} \sum_{i=0}^{j} a_i^{j-i}$ of the double sum (3) throughout the article.

2 Main results

Let us put

$$g(\delta) = \tfrac{1-f(\delta)}{1+f(\delta)}, \; h(\delta) = \left(1 - \tfrac{4\delta^2}{1+f(\delta)}\right)^{-2}, \; p(\delta) = \tfrac{2\delta}{1+f(\delta)}.$$

As there will be no possibility of misunderstanding we will use f, g, h, p, ϕ_i, $\Phi_{i,j}$, C_j instead of $f(\delta)$, $g(\delta)$, $h(\delta)$, $p(\delta)$, $\phi_i(\delta)$, $\Phi_{i,j}(\delta)$, $C_j(\delta)$ just for simplification of the notations.

We will use the notation $[x]$ to present the integer part of x.

Theorem 2. *For every* $\delta \in \left(0, \frac{2}{5}\right]$ *inequality* $L(\delta) < 1$ *holds.*

Theorem 3. *Let* $\delta \in \left[0, \frac{1}{2}\right)$ *and* $\varepsilon > 0$. *If* $N = \left[\dfrac{\log\left(\frac{\varepsilon(1-p)^2(1-\delta)^4}{4h}\right)}{\log p}\right] + 1$ *and* $M = \left[\dfrac{\mathrm{LambertW}\left(\frac{\varepsilon(1-\delta)^2}{2h} p(1-p)^2 \log p\right)}{\log p}\right] - 1$ *then for every* $n \geq N$ *and* $m \geq M$

inequality

$$\left| L(\delta) - \frac{\displaystyle\sum_{j=0}^{m}\sum_{i=0}^{j} \frac{(-\delta)^j \Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}}{\left(1+\sum_{i=1}^{n}\frac{(-\delta)^i}{C_i(\delta)}\right)^2} \right| < \varepsilon$$

holds.

3 Auxiliary results

It is easy to see that

$$g : [0, 1/2) \to [0, 1) \text{ and } p : [0, 1/2) \to [0, 1). \tag{4}$$

We will need in the sequel the fraction $\frac{C_{j-1}}{C_j}$. From $C_j = \frac{(1+f)^{j+1}}{2^{j+1}f}\left(1 - g^{j+1}\right)$ we get

$$\frac{C_{j-1}}{C_j} = \frac{2(1-g^j)}{(1+f)(1-g^{j+1})}. \tag{5}$$

Lemma 1. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and for every $j = 0, 1, 2, \ldots$ inequality $\frac{\delta C_j(\delta)}{C_{j+1}(\delta)} \leq 1$ holds.*

Proof. From (5) and (4) we get that the inequality $\frac{\delta C_j(\delta)}{C_{j+1}(\delta)} \leq p(\delta) < 1$ holds for every $\delta \in \left[0, \frac{1}{2}\right)$. □

Lemma 2. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and for every $j = 1, 2, \ldots$ inequality $\frac{\delta^j}{C_j(\delta)} \leq p^j(\delta)$ holds.*

Proof. The inequality

$$\frac{\delta^j}{C_j(\delta)} = \left(\frac{2\delta}{1+f(\delta)}\right)^j \cdot \frac{2f(\delta)}{(1+f(\delta))\,(1-g^{j+1}(\delta))} \leq \left(\frac{2\delta}{1+f(\delta)}\right)^j = p^j(\delta)$$

is true for every $\delta \in \left[0, \frac{1}{2}\right)$. □

Lemma 3. *For every $\delta \in \left[0, \frac{1}{2}\right)$ the series*

$$\sum_{j=1}^{\infty} \frac{\delta^{2j}}{C_{2j}(\delta)} \tag{6}$$

and

$$\sum_{j=1}^{\infty} \frac{\delta^{2j-1}}{C_{2j-1}(\delta)} \tag{7}$$

are convergent.

Proof. From Lemma 2, the convergence of the series $\sum_{j=1}^{\infty} p^{2j}$ and $\sum_{j=1}^{\infty} p^{2j-1}$ it follows that the series (6) and (7) are convergent. □

By Lemma 3 it follows that the series

$$\sum_{j=0}^{\infty} \frac{(-\delta)^j}{C_j(\delta)}$$

is absolutely convergent and consequently there holds the representation

$$\begin{aligned} S_1(\delta) &= \left(1 + \sum_{j=1}^{\infty} \frac{\delta^{2j}}{C_{2j}(\delta)} - \sum_{j=1}^{\infty} \frac{\delta^{2j-1}}{C_{2j-1}(\delta)}\right)^2 \\ &= \left(1 + \sum_{j=1}^{\infty} \frac{(-\delta)^j}{C_j(\delta)}\right)^2 = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-\delta)^{j+i}}{C_j(\delta)C_i(\delta)} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{L_j(\delta)L_i(\delta)}{L_0(\delta)L_0(\delta)} = \frac{\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} L_j(\delta)L_i(\delta)}{L_0(\delta)L_0(\delta)} \end{aligned} \tag{8}$$

for every $\delta \in \left[0, \frac{1}{2}\right)$.

Lemma 4. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and every $\varepsilon > 0$ there exist $N_1 = N_1(\delta, \varepsilon) \in \mathbb{N}$ and $M_1 = M_1(\delta, \varepsilon) \in \mathbb{N}$ such that for any $n \geq N_1$ inequality $\left|\frac{(-\delta)^n}{C_n(\delta)}\right| < \varepsilon$ holds and for any $m \geq M_1$ inequality $\frac{(2+m)p^{m+1}(\delta)}{(1-p(\delta))^2} < \varepsilon$ holds.*

Proof. Let us put $N_1 \geq \left[\frac{\log \varepsilon}{\log p}\right] + 1$. From Lemma 1 and Lemma 2 it follows that the inequality $\left|\frac{(-\delta)^n}{C_n(\delta)}\right| \leq \left|\frac{(-\delta)^{N_1}}{C_N(\delta)}\right| \leq p^{N_1}(\delta) < p^{\frac{\log(\varepsilon)}{\log p}}(\delta) = \varepsilon$ holds true for every $n \geq N_1$.

Let us put $M_1 = \left[\frac{\mathrm{LambertW}\left(\varepsilon \mathrm{p}(1-\mathrm{p})^2 \log \mathrm{p}\right)}{\log p}\right] - 1$. For any fixed $\delta \in [0, 1/2)$ the function $F(u) = (2+u)p^{u+1}(\delta) : [M_1, +\infty)$ is a decreasing function. Therefore the inequality $\frac{(2+m)p^{m+1}}{(1-p)^2} \leq \frac{(2+M_1)p^{M_1+1}}{(1-p)^2} < \varepsilon$ holds true for every $m \geq M_1$. □

Lemma 5. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and every $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$, such that the inequality $\left|\sum_{j=0}^{\infty} \frac{(-\delta)^j}{C_j(\delta)} - \sum_{j=0}^{n-1} \frac{(-\delta)^j}{C_j(\delta)}\right| < \varepsilon$ holds for every $n \geq N_2$.*

Proof. Let us put $N_2 \geq \left[\frac{\log \varepsilon}{\log p}\right] + 1$. From Lemma 1 the sequence $\left|\frac{(-1)^j \delta^j}{C_j(\delta)}\right|$ is decreasing and consequently from the convergence of the series $\sum_{j=0}^{\infty} \frac{(-\delta)^j}{C_j(\delta)}$ and Lemma 4 it follows that the inequality

$$\left|\sum_{j=0}^{\infty} \frac{(-\delta)^j}{C_j(\delta)} - \sum_{j=0}^{n-1} \frac{(-\delta)^j}{C_j(\delta)}\right| = \left|\sum_{j=n}^{\infty} \frac{(-\delta)^j}{C_j(\delta)}\right| \leq \frac{\delta^n}{C_n(\delta)} < \frac{\delta^{N_2}}{C_{N_2}(\delta)} < \varepsilon$$

holds for every $n \geq N_2$. □

Lemma 6. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and for every $j = 0, 1, 2, \dots$ the inequality*

$$\phi_j(\delta) \leq \phi_{j+1}(\delta). \tag{9}$$

holds.

Proof. From (5) it follows that the inequality $\frac{\phi_j}{\phi_{j+1}} = \frac{C_{j-1}C_{j+1}}{(C_j)^2} = \frac{(1-g^j)(1-g^{j+2})}{(1-g^j)^2} = \frac{(1-g^j)^2 - g^j(g-1)^2}{(1-g^j)^2} < 1$ holds for every $\delta \in \left(0, \frac{1}{2}\right)$.

For $\delta = 0$ inequality (9) is trivial, because $\phi_j(0) = 0$. □

Lemma 7. *For every $\delta \in \left[0, \frac{1}{2}\right)$ there holds $\lim_{j\to\infty} \phi_j(\delta) = \frac{2\delta^2}{1+f(\delta)}$.*

Proof. From (5) we get that

$$\lim_{j\to\infty} \phi_j(\delta) = \lim_{j\to\infty} \frac{\delta^2 C_j(\delta)}{C_{j+1}(\delta)} = \lim_{j\to\infty} \frac{2\delta^2}{1+f(\delta)} \left(\frac{1-g^j(\delta)}{1-g^{j+1}(\delta)}\right) = \frac{2\delta^2}{1+f(\delta)}.$$

holds for every $\delta \in \left[0, \frac{1}{2}\right)$. □

Lemma 8. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and every $i, j = 0, 1, 2, \dots$ the inequality $\Phi_{i,j}(\delta) \leq h(\delta)$ holds.*

Proof. From Lemma 6 and Lemma 7 we get the inequality

$$\Phi_{i,j}(\delta) < \left(1 - 2\lim_{j\to\infty} \phi_j(\delta)\right)^{-2} = \left(1 - \frac{4\delta^2}{1+f(\delta)}\right)^{-2} = h(\delta).$$

holds true for every $\delta \in \left[0, \frac{1}{2}\right)$. □

Lemma 9. *For every $\delta \in \left[0, \frac{1}{2}\right)$ the double series*

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-\delta)^{i+j}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)}. \tag{10}$$

is absolutly convergent.

Proof. From Lemma 2 and Lemma 8 we have that for every $\delta \in \left[0, \frac{1}{2}\right)$ the inequality $\left|\frac{(-\delta)^{i+j}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)}\right| \leq p^{j+i}(\delta)h(\delta)$ holds.

The double series $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} p^{i+j}$ is convergent for every $p \in [0, 1)$ and consequently the series (10) is absolutely convergent. □

Lemma 10. *For every $\delta \in \left[0, \frac{1}{2}\right)$ the inequality*

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \left|\frac{(-\delta)^{i+j}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)}\right| \leq \frac{h(\delta)}{(1-p(\delta))^2} \tag{11}$$

holds.

Proof. From Lemma 9 the series (11) is absolutely convergent. Therefore we can change the summation without changing the sum. Thus we get that the the inequality

$$\begin{aligned} S_2(\delta) &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left|\frac{(-\delta)^{i+j}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)}\right| = \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{(-\delta)^{j}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}\right| \\ &\le h\sum_{j=0}^{\infty}\sum_{i=0}^{j}p^j = h\sum_{j=0}^{\infty}(j+1)p^j = \frac{h}{(1-p)^2} \end{aligned}$$

holds for every $\delta \in \left[0, \frac{1}{2}\right)$. □

Lemma 11. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and every $\varepsilon > 0$ there exists $M_2 \in \mathbb{N}$ such that for every $m \ge M_2$ the inequality*

$$J_m(\delta) = \left|\sum_{j=m+1}^{\infty}\sum_{i=0}^{j}\frac{(-\delta)^{j}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}\right| < \varepsilon$$

holds.

Proof. Let us put $M_2 = \left[\frac{\mathrm{LambertW}\left(\frac{\varepsilon}{h}\mathrm{p}(1-\mathrm{p})^2\log\mathrm{p}\right)}{\log p}\right] - 1$.

From Lemma 2, Lemma 8 and Lemma 4 we get that the inequality

$$\begin{aligned} J_m(\delta) &= \left|\sum_{j=m+1}^{\infty}\sum_{i=0}^{j}\frac{(-\delta)^{j}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}\right| \le h\left|\sum_{j=m+1}^{\infty}(j+1)p^j\right| \\ &= h\frac{(m+2)p^{m+1}-(m+1)p^{m+2}}{(1-p)^2} \le h\frac{(2+m)p^{m+1}}{(1-p)^2} < \varepsilon \end{aligned}$$

holds for every $m \ge M_2$, because $(m+2)p^{m+1} - (m+1)p^{m+2} > 0$ for every $m \ge M_2$. □

Lemma 12. *For every $\delta \in \left[0, \frac{1}{2}\right)$ and every $\varepsilon > 0$ there exists $M_2 \in \mathbb{N}$ such that for every $m \ge M_2$ the inequality*

$$\left|\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\delta)^{j+i}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)} - \sum_{j=0}^{m}\sum_{i=0}^{j}\frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}\right| < \varepsilon$$

holds.

Proof. By Lemma 9 the series $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\delta)^{j+i}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)}$ is absolutely convergent and therefore it follows that we can change the summation in it without changing the sum

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\delta)^{j+i}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)} = \sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}.$$

From Lemma 11 we get that the inequality

$$\begin{aligned} B_m(\delta) &= \left| \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-\delta)^{j+i}\Phi_{i,j}(\delta)}{C_j(\delta)C_i(\delta)} - \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)} \right| \\ &= \left| \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)} - \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)} \right| \\ &= \left| \sum_{j=m+1}^{\infty} \sum_{i=0}^{j} \frac{(-\delta)^{j+i}\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)} \right| < \varepsilon \end{aligned}$$

holds for every $m \geq M_2 = \left[\frac{\mathrm{LambertW}\left(\frac{\varepsilon \mathrm{p}(1-\mathrm{p})^2}{\mathrm{h}} \log \mathrm{p}\right)}{\log p} \right] - 1.$ □

The inequalities (12) and (13) hold for every $\delta \in \left[0, \frac{1}{2}\right)$

$$\frac{\delta C_j(\delta)}{C_{j+1}(\delta)} \leq \frac{2\delta}{1+f(\delta)}, \tag{12}$$

$$\delta^2 \leq \phi_j(\delta) \leq \frac{2\delta^2}{1+f(\delta)}. \tag{13}$$

We will need the equality $(1+f(\delta))(1+g(\delta)) = 2$.

Lemma 13. *For every $\delta \in \left(0, \frac{2}{5}\right]$ and any $j, i = 1, 2, \ldots,$ the inequalities*

$$\frac{\delta^{j-1}}{C_{j-1}(\delta)} \left(\Phi_{i,j-1}(\delta) - 1\right) > \frac{\delta^j}{C_j(\delta)} \left(\Phi_{i,j}(\delta) - 1\right), \tag{14}$$

$$\frac{\delta^{j-1}}{C_{j-1}(\delta)} \left(\Phi_{0,j-1}(\delta) - 1\right) > \frac{2\delta^j}{C_j(\delta)} \left(\Phi_{0,j}(\delta) - 1\right). \tag{15}$$

hold

Proof. Let us put $c_1(\delta) = \dfrac{4\delta^2}{(1-2\delta^2)^2 (1+f(\delta))(1+f(\delta)-4\delta^2)^2}$. and

$$b_1(\delta) = (1-\delta^2)(1+f(\delta))\left(1+f(\delta)-4\delta^2\right)^2 - 4\delta(1-2\delta^2)^2(1+f(\delta)-2\delta^2).$$

For any $\delta \in (0, 0.4)$ the inequalities $c_1(\delta) > 0$ and $b_1(\delta) > 0$ hold. From the inequalities

$$\begin{aligned} A_1(\delta) &= \left(\Phi_{i,j-1}(\delta) - 1\right) - \tfrac{\delta C_{j-1}(\delta)}{C_j(\delta)} \left(\Phi_{i,j}(\delta) - 1\right) \\ &\geq \left(\left(1-2\delta^2\right)^{-2} - 1\right) - \tfrac{\delta C_{j-1}(\delta)}{C_j(\delta)} \left(\left(1 - \tfrac{4\delta^2}{1+f(\delta)}\right)^{-2} - 1\right) \\ &\geq \left(\tfrac{4\delta^2(1-\delta^2)}{(1-2\delta^2)^2} - \tfrac{16\delta^3}{1+f(\delta)} \cdot \tfrac{1+f(\delta)-2\delta^2}{(1+f(\delta)-4\delta^2)^2}\right) = c_1(\delta)b_1(\delta) \end{aligned}$$

we get that $A_1(\delta) > 0$ for $\delta \in (0, 0.4]$ and consequently (14) holds true.

Let us put $c_2(\delta) = \dfrac{\delta^2}{(1-\delta^2)^2\,(1+f(\delta))\,(1+f(\delta)-2\delta^2)^2}$ and

$$b_2(\delta) = (2-\delta^2)(1+f(\delta))\left(1+f(\delta)-2\delta^2\right)^2 - 4\delta(1-\delta^2)^2(1+f(\delta)-\delta^2).$$

For any $\delta \in (0, 0.4)$ the inequalities $c_2(\delta) > 0$ and $b_2(\delta) > 0$ hold.

From $\phi_0(\delta) = 0$ we have the equalities

$$\begin{aligned} A_2(\delta) &= (\Phi_{0,j-1}(\delta) - 1) - \tfrac{2\delta C_{j-1}(\delta)}{C_j(\delta)}\,(\Phi_{0,j}(\delta) - 1) \\ &= (\Phi_{0,j-1}(\delta) - 1) - \tfrac{2\delta C_{j-1}(\delta)}{C_j(\delta)}\,(\Phi_{0,j}(\delta) - 1) \\ &= \tfrac{\delta^2(2-\delta^2)}{(1-\delta^2)^2} - \tfrac{4\delta^3}{1+f(\delta)} \cdot \tfrac{1+f(\delta)-\delta^2}{(1+f(\delta)-2\delta^2)^2} = c_2(\delta) b_2(\delta) \end{aligned}$$

Thus we get that $A_2(\delta) > 0$ for $\delta \in (0, 0.4]$ and consequently (15) holds. □

4 Proof of main results

Just for some simplification of the notations we will denote:

$$\overline{L}_0(\delta) = L_0(\delta), \quad \overline{L}_i(\delta) = \frac{L_i(\delta)}{L_0(\delta)} = \frac{(-\delta)^i}{C_i(\delta)}, \quad j \in \mathbb{N}.$$

Proof. (of Theorem 2) From $\overline{L}_0(\delta)\overline{L}_0(\delta)\left(\Phi_{0,0}(\delta) - 1\right) = 0$ and the absolute convergence of the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i(\delta)\overline{L}_j(\delta)$ and $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i(\delta)\overline{L}_j(\delta)\Phi_{i,j}(\delta)$ it follows that we can change the summation in the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i(\delta)\overline{L}_j(\delta)\Phi_{i,j}(\delta) - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i(\delta)\overline{L}_j(\delta)$$

without changing the sum. From the chain of equalities

$$\begin{aligned} S(\delta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i\overline{L}_j\Phi_{i,j} - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i\overline{L}_j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i\overline{L}_j\left(\Phi_{i,j} - 1\right) \\ &= \sum_{i=1}^{\infty}\left(\sum_{j=0}^{2i-1} \overline{L}_j\overline{L}_{2i-1-j}\left(\Phi_{i,2i-1-j} - 1\right) + \sum_{j=0}^{2i} \overline{L}_j\overline{L}_{2i-j}\left(\Phi_{i,2i-j} - 1\right)\right) \\ &= \sum_{i=1}^{\infty} \overline{L}_0\left(\overline{L}_{2i-1}\left(\Phi_{0,2i-1} - 1\right) + 2\overline{L}_{2i}\left(\Phi_{0,2i} - 1\right)\right) \\ &\quad + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{2i-1} \overline{L}_j\overline{L}_{2i-1-j}\left(\Phi_{i,2i-1-j} - 1\right) + \sum_{j=1}^{2i-1} \overline{L}_j\overline{L}_{2i-j}\left(\Phi_{i,2i-j} - 1\right)\right) \end{aligned}$$

and Lemma 13 we obtain that $S(\delta) < 0$ for every $\delta \in \left(0, \frac{2}{5}\right]$. Thus using the representation (8) we get that the inequality

$$\begin{aligned} L(\delta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} L_i(\delta)L_j(\delta)\Phi_{i,j}(\delta) = \frac{\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i(\delta)\overline{L}_j(\delta)\Phi_{i,j}(\delta)}{S_1(\delta)} \\ &= \frac{\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i\overline{L}_j\Phi_{i,j}(\delta)}{\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{L}_i\overline{L}_j} < 1 \end{aligned} \tag{16}$$

holds for every $\delta \in \left(0, \frac{2}{5}\right]$. □

Proof. (of Theorem 3) From the result that the series

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \overline{L}_j\overline{L}_i\Phi_{i,j} \quad \text{and} \quad 1 + \sum_{j=1}^{\infty} \frac{\delta^{2j}}{C_{2j}(\delta)} - \sum_{j=1}^{\infty} \frac{\delta^{2j-1}}{C_{2j-1}(\delta)}$$

are absolutely convergent it follows that their sums do not change if we change the summation. We will use the representations

$$S_1 = 1 - \frac{\delta}{C_1(\delta)} + \sum_{j=1}^{\infty}\left(\frac{\delta^{2j}}{C_{2j}(\delta)} - \frac{\delta^{2j+1}}{C_{2j+1}(\delta)}\right)$$

and

$$S_2 = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \overline{L}_j\overline{L}_i\Phi_{i,j} = \sum_{j=0}^{\infty}\sum_{i=0}^{j} \overline{L}_i\overline{L}_{j-i}\Phi_{i,j-i}.$$

From Lemma 1 and the equality $C_1(\delta) = 1$ inequality $1 - \delta \le S_1 \le 1$ follows.

Let us denote $J_m = \sum_{j=0}^{m}\sum_{i=0}^{j} L_iL_{j-i}\Phi_{i,j-i}$ and $K_n(\delta) = \sum_{j=0}^{n-1} \frac{(-\delta)^j}{C_j(\delta)}$.

Let $\varepsilon > 0$ be arbitrarily chosen. From Lemma 11 it follows that there exist

$$M = \left[\frac{\mathrm{LambertW}\left(\frac{\varepsilon(1-\delta)^2}{2\mathrm{h}}(1-\mathrm{p})^2\,\mathrm{p}\log\mathrm{p}\right)}{\log p}\right] - 1,$$

such that the inequality

$$|S_2 - J_m| = \left|S_2 - \sum_{j=0}^{m}\sum_{i=0}^{j} L_iL_{j-i}\Phi_{i,j-i}\right| < \frac{\varepsilon}{2}(1-\delta)^2 \tag{17}$$

holds for every $m \ge M$. Similarly from Lemma 5 it follows that there exist

$$N = \left[\frac{\log\left(\frac{\varepsilon(1-p)^2(1-\delta)^4}{4h}\right)}{\log p}\right] + 1,$$

such that the inequality

$$|S_1 - K_n| = \left| S_1 - \sum_{j=0}^{n-1} \frac{(-\delta)^j}{C_j(\delta)} \right| < \frac{\varepsilon\,(1-p)^2\,(1-\delta)^4}{4h} \tag{18}$$

holds for every $n \geq N$.

From (18) we get that the inequality

$$|S_1^2 - K_n^2| = |S_1 + K_n|.|S_1 - K_n| \leq 2|S_1|.|S_1 - K_n| < \frac{\varepsilon\,(1-p)^2\,(1-\delta)^4}{2h}$$

holds for every $n \geq N$. From (17) we get that the inequality

$$\left| \frac{S_2 - J_m}{S_1^2} \right| \leq \frac{\frac{\varepsilon}{2}\,(1-\delta)^2}{(1-\delta)^2} = \frac{\varepsilon}{2}$$

holds for every $m \geq M$. From Lemma 10 we have the inequality

$$|J_m| \leq \sum_{j=0}^{m} \sum_{i=0}^{j} |L_i L_{j-i} \Phi_{i,j-i}| \leq \frac{h}{(1-p)^2}$$

and therefore we get the inequality $|J_m| \left| \frac{K_n^2 - S_1^2}{S_1^2 K_n^2} \right| = \frac{|J_m|}{S_1^2 K_n^2} \left| K_n^2 - S_1^2 \right| \leq \frac{\varepsilon}{2}$, where we use the inequality $S_1(\delta) K_n(\delta) \geq (1-\delta)^2$. Consequently the inequality

$$\left| \frac{S_2}{S_1^2} - \frac{J_m}{K_n^2} \right| \leq \left| \frac{S_2}{S_1^2} - \frac{J_m}{S_1^2} \right| + \left| \frac{J_m}{S_1^2} - \frac{J_m}{K_n^2} \right| = \left| \frac{S_2 - J_m}{S_1^2} \right| + |J_m| \left| \frac{K_n^2 - S_1^2}{S_1^2 K_n^2} \right| < \varepsilon$$

holds for every $m \geq M$ and $n \geq N$. □

Corollary 1. *The correction coefficient (1) is a continuous function in every closed interval* $[0, a] \subset [0, 1/2)$.

Proof. From the representation (16) it follows that the correction coefficient is a fraction of two series. By Lemma 3 and Lemma 9 it follows that the correction coefficient is a fraction of two continuous functions in every closed interval $[0, a] \subset [0, 1/2)$ and therefore it is a continuous function in every closed interval $[0, a] \subset [0, 1/2)$. □

5 Applications

Formula (2) presents the force of electrostatic interaction between two spheres. Therefore it is interesting to know the values of the coefficient (1) for different values of $\delta = \frac{r}{R}$, where r is the radii of the spheres and R is the distance between their centers. The functions which define the series $L(\delta)$ are complicated and therefore we could not calculate the precise values of $L(\delta)$, for any $\delta \in (0, 1/2)$. Theorem 3 however, gives us the possibility to calculate $L(\delta)$ with an arbitrary precision $\varepsilon > 0$, by calculating the partial sums

$$\frac{\displaystyle\sum_{j=0}^{M}\sum_{i=0}^{j}\frac{(-\delta)^j\Phi_{i,j-i}(\delta)}{C_i(\delta)C_{j-i}(\delta)}}{\left(1+\displaystyle\sum_{i=1}^{N}\frac{(-\delta)^i}{C_i(\delta)}\right)^2} \tag{19}$$

for any $\delta \in [0, 1/2)$.

Using Theorem 3 we get the values $\max\{M, N\}$ for some $\delta \in \left(0, \frac{1}{2}\right)$ and $\varepsilon > 0$ in Table 1.

Table 1: Values of $\max\{M, N\}$, depending on δ and ε

$\varepsilon \setminus \delta$	0.0001	0.001	0.1	0.2	0.3	0.4	0.49	0.499	0.4999	0.49999
10^{-1}	1	1	2	4	6	13	75	336	1363	5250
10^{-2}	1	1	3	5	8	16	88	374	1482	5625
10^{-3}	1	2	4	7	10	20	100	412	1601	5999
10^{-4}	2	2	5	8	13	23	112	450	1720	6373
10^{-5}	2	2	6	10	15	27	124	487	1838	6746
10^{-6}	2	3	7	11	17	30	135	525	1956	7119
10^{-7}	2	3	8	13	19	34	147	562	2074	7491
10^{-8}	3	3	9	14	22	37	159	600	2192	7862
10^{-9}	3	4	10	16	24	41	171	637	2310	8234
10^{-10}	3	4	11	17	26	44	183	674	2428	8605

It is seen from Table 1 that, when δ is relatively small (i.e. the distance between the spheres is relatively large), the upper limits of summation indexes in the partial sums (19) are small too. However, if the two spheres are at a relatively small distance between one another, the upper limits of summation indexes in the partial sums (19) are sharply increasing. A similar observation with a different technique is obtained in [13].

Using Theorem 3 we get the values $L(\delta)$ for some $\delta \in (0, 1/2)$ and $\varepsilon > 0$.

From Table 2 we see, that the values of $L(\delta)$ are close to one when the radii of the spheres are relatively small when compared with the distance between their centers. From Table 2 and Theorem 2 it follows that the Coulomb's law gives relatively accurate results in case that the radius r of the spheres is at least an order of magnitude smaller than the distance R between their centres.

From Table 2 we see, that the values of $L(\delta)$ are relatively smaller than one when the two spheres are at a relatively small distance between one another. This shows that the force F of interaction between the spheres highly differs from the Coulomb force F_C, when the spheres are at a relatively small distance between

Table 2: Approximate values of $L(\delta)$, depending on δ and ε

$\varepsilon \setminus \delta$	0.0001	0.1	0.3	0.49	0.49999
10^{-1}	1.0	1.0	0.89	0.63	0.61
10^{-2}	1.00	0.996	0.889	0.628	0.615
10^{-3}	1.000	0.9960	0.8890	0.6279	0.6149
10^{-4}	1.0000	0.99595	0.88901	0.62792	0.61491
10^{-5}	1.00000	0.995954	0.889009	0.627922	0.614915

one another. A similar observation for the behavior of $L(\delta)$ but with a different technique is obtained in [13].

This observation raises the following open questions:

- $L(\delta) < 1$ for any $\delta \in [0, 1/2)$,
- $L(\delta)$ is a decreasing function in $[0, 1/2)$,
- the limit $\lim_{\delta \to 1/2} L(\delta)$ exists and is smaller than 1,

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