## Smoothness in Musielak–Orlicz Sequence Spaces

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ABSTRACT: Estimates for the order of Fréchet differentiability of norms in Musielak–Orlicz sequence spaces and bump functions in Nakano sequence spaces are found.

**Keywords:** Musielak–Orlicz sequence spaces, Nakano sequence spaces, Fréchet differentiability.

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## 1 Preliminaries

In the next $X$, $Y$ are Banach spaces, $\mathbb{N}$ the naturals, $\mathbb{R}$ the reals. Everywhere differentiability is understood as Fréchet differentiability. Throughout this paper, a bump function or simply a bump in $X$ is a non–zero real–valued function on $X$ with bounded support. The class of all $k$–times differentiable (continuously differentiable) real functions in $U \subset X$ is denoted by $F^k(U)$ $(C^k(U))$.

Let us recall that an Orlicz function $M$ is an even, continuous, nondecreasing, convex function defined for $t \geq 0$ so that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. A sequence $\varphi = \{\varphi_n\}_{n=1}^{\infty}$ of Orlicz functions is called a Musielak–Orlicz function. Throughout the paper we shall write $M$ for an Orlicz function and $\varphi$, $\psi$ for Musielak–Orlicz functions. The Musielak–Orlicz sequence spaces $\ell_\varphi$, generated by $\varphi$ is the set of all real sequence $x = \{x_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n(\lambda x_n) < \infty$ for some $\lambda > 0$. The Luxemburg norm is defined by $\|x\|_\varphi = \inf\{r > 0 : \sum_{n=1}^{\infty} \varphi_n(x_n/r) \leq 1\}$.

Let $1 \leq p_n$, $n \in \mathbb{N}$ be a sequence of reals. The Musielak–Orlicz sequence space $\ell_\varphi$, where $\varphi = \{\frac{t^{p_n}}{p_n}\}_{n=1}^{\infty}$ is called Nakano sequence space $\ell(\{p_n\})$.

The Orlicz function $M$ is said to have the property $\Delta_2$ at 0 if there exists a constant $c$ such that $M(2t) \leq cM(t)$ for every $t \in [0,1]$. A Musielak–Orlicz function $\varphi$ satisfies the $\delta_2$ condition if there are positive constants $K$, $\delta$ and a nonnegative sequence $\{c_n\}$ in $\ell_1$ such that for each $n \in \mathbb{N}$ the condition $\varphi(x) \leq \delta$ implies $\varphi_n(2x) \leq K\varphi_n(x) + c_n$

We say that two Orlicz functions $M$ and $N$ are equivalent at 0 if $c^{-1}M(c^{-1}t) \leq N(t) \leq cM(ct)$ for some constant $c > 0$ and for every $t \in [0,1]$ and write $M \sim N$.

To every Orlicz function $M$ the following number is associate (see [5], p. 143)

$$\alpha_M = \sup\{p : \sup_{0<\lambda,t\leq 1} M(\lambda t)/(M(\lambda)t^q) < \infty\}$$
$$\beta_M = \inf\{p : \inf_{0<\lambda,t<1} M(\lambda t)/(M(\lambda)t^q) > 0\}$$

and to every Musielak–Orlicz function $\varphi = \{\varphi_n\}_{n=1}^{\infty}$ we associate the numbers

$$\alpha_\varphi = \liminf_{n\to\infty} \alpha_{\varphi_n} \quad \beta_\varphi = \limsup_{n\to\infty} \beta_{\varphi_n}$$

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## 2 Smooth renormings in Musielak–Orlicz sequence spaces

**Lemma 2.1** *Let $\varphi = \{\varphi_n\}_{n=1}^{\infty}$ be a Musielak–Orlicz function. There exist a Musielak–Orlicz function $\Psi = \{\Psi_n\}_{n=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ such that $\Psi_n$ is infinitely many times differentiable for every $n \in \mathbb{N}$ and*

(i) $\varphi_n \sim \Psi_n$ *at* $0$

(ii) $t^k|\Psi_n^{(k)}(t)| \leq c_k \Psi_n(c_k t),\ t \in [0,\infty),\ c_k > 0,\ k = 1, 2, \ldots$

*for every $n$.*

**Proof:** Following [6] we define a Musielak–Orlicz functions $\psi = \{\psi_n\}_{n=1}^{\infty}$, $\Psi = \{\Psi_n\}_{n=1}^{\infty}$ by $\psi_n(t) = \int_0^t \frac{\varphi(u)}{u} du$ and $\Psi_n(t) = \int_0^t \frac{\psi_n(u)}{u} \exp \frac{u}{u-t} du$. It is easily seen that $\varphi_n \sim \psi_n \sim \Psi_n$ at 0. To varify (ii) it is enough to write:

$$\Psi_n^{(k)}(t) = \int_0^t \frac{\psi(u)}{u} \frac{d^k}{dt^k} \exp \frac{u}{u-t} du = \int_0^t \psi(u) \sum_{j=0}^{k-1} c_j(k)(u-t)^{j-2k} u^{k-j-1} \exp \frac{u}{u-t} du,$$

where $c_j(k) \in \mathbb{N}$, $j = 0, 1, \ldots, k-1$. After a substitution $u = vt$ in the last integral we obtain the estimate for $t \in [0, \infty)$

$$t^k |\Psi_n^{(k)}(t)| \leq c(k) \Psi_n(t),$$

where the constants $c(k) = \int_0^1 \sum_{j=0}^{k-1} c_j(k)(1-v)^{j-2k} v^{k-j-1} exp \frac{v}{v-1} dv.$ □

We shall need a sufficient condition for isomorphism of Musielak–Orlicz sequence spaces.

**Lemma 2.2** *Let $\varphi = \{\varphi_n\}$ and $\psi = \{\psi_n\}$ be Musielak–Orlicz functions. If there exist positive constants $\delta_1$, $\delta_2$, $K_1$, $K_2$, $L_1$, $L_2$ and sequences of nonnegative numbers $\{a_n\}$, $\{b_n\}$ from $\ell_1$, such that for every $n \in \mathbb{N}$ are fulfilled*

$$\psi_n(t) \leq K_1 \varphi_n(K_2 t) + a_n \quad if \quad \varphi_n(t) < \delta_1, \quad t \geq 0$$

$$\varphi_n(t) \leq L_1 \psi_n(L_2 t) + b_n \quad if \quad \varphi_n(t) < \delta_2, \quad t \geq 0$$

*the spaces $\ell_\varphi$ and $\ell_\psi$ are isomorphic.*

**Proof:** According to [7] (Theorem 8.11) follows that $\ell_\psi$ and $\ell_\varphi$ coincide as sets. So it is sufficient to prove that there are constants $a, b > 0$ such that $a\|x\|_\varphi \leq \|x\|_\psi \leq b\|x\|_\varphi$.

WLOG we may assume $K_1, L_1 > 1$. The inequality

$$\sum_{n=1}^{\infty} \psi_n \left( \frac{x_n}{K_1 \left(1+\sum_{i=1}^{\infty} a_i\right) K_2 \|x\|_\varphi} \right) \leq \frac{1}{K_1 \left(1+\sum_{i=1}^{\infty} a_i\right)} \sum_{n=1}^{\infty} \psi_n \left( \frac{x_n}{K_2 \|x\|_\varphi} \right)$$
$$\leq \frac{1}{K_1 \left(1+\sum_{i=1}^{\infty} a_i\right)} K_1 \sum_{n=1}^{\infty} \left( \varphi_n \left( \frac{x_n}{\|x\|_\varphi} \right) + a_n \right)$$

give us $\|x\|_\psi \leq (K_1 + \sum_{i=1}^{\infty} a_i) K_2 \|x\|_\varphi$. In the same way is obtained $\|x\|_\varphi \leq (L_1 + \sum_{i=1}^{\infty} b_i) L_2 \|x\|_\psi$. $\square$

For $p > 0$ define $E(p) = \begin{cases} p-1 & p \in \mathbb{N}, \\ [p] & p \notin \mathbb{N}. \end{cases}$ Using Lemma 2.1 and 2.2 and following [6] one can prove

**Theorem 2.1** *Let $\varphi$ be a Musielak–Orlicz function and $1 \leq k = E(\alpha_\varphi)$ Then there exists an equivalent $k$–times Fréchet differentiable norm in $h_\varphi$. If in addition $\beta_\varphi < \infty$ then there exists an equivalent $k$–times uniformly Fréchet differentiable norm in $\ell_\varphi$.*

**Corollary 2.1** *Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of reals such that $\underline{p} = \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n = \overline{p}$ and $1 \leq \underline{p} \leq \overline{p} \leq \infty$. Then there exists an equivalent norm in $h(\{p_n\})$, which is $E(\underline{p})$–times differentiable. If in addition $\overline{p} < \infty$ then there is an equivalent norm in $\ell(\{p_n\})$, which is $E(\underline{p})$–times uniformly differentiable. Especially if $\lim_{n\to\infty} p_n = \infty$ then there is an infinitly many times Fréchet diferentiable equivalent norm in $h(\{p_n\})$.*

## 3 Differentiability of bumps in $\ell(\{p_n\})$.

It is well known that if in a Banach space there is an equivalent $k$–times differentiable norm, then there exist a bump with the same order of differentiability. That is why the negative results on differentiability of bumps give negative results for the differentiability in the class of all equivalent norms.

Denote $J_n = \{-1, +1\}^n$. $X$ is said to be of cotype $q \in [2, \infty)$ iff there is a constant $C$ s.t. for all finite subsets $(x_1, \ldots, x_n)$ of $X$ is fulfilled:

$$\left( \sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq \frac{C}{2^n} \sum_{\varepsilon_i \in J_n} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|.$$

We say that $X$ is saturated with subspaces with the property (*) if in every infinite dimensional subspace $Z$ of $X$ there is an infinite dimensional subspace $Y$ of $Z$ isomorphic to a space with the property (*).

A necessary and sufficient condition for coincidence as sets of two Nakano sequence spaces in [2]. Lemma 2.2 and the proof of Corollary 2.5 in [2] gives us the following necessary and sufficient condition for the isomorphism of two Nakano sequence spaces.

**Lemma 3.1** *Let $1 \leq p_n$, $n \in \mathbb{N}$. The Nakano sequence space $\ell(\{p_n\})$ is isomorphic to $\ell_p$ iff there exists a constant $0 < C < 1$ s.t. $\sum_{p_n \neq p} C^{\frac{1}{|p_n - p|}} < \infty$.*

**Lemma 3.2** *(Theorem 3.3 [2]) Let $2 \leq q < \infty$ and $1 \leq p_n < \infty$, $n \in \mathbb{N}$. Then $\ell(\{p_n\})$ has cotype $q$ iff there exists $0 < C < 1$ such that $\sum_{p_n > q} C^{\frac{1}{p_n - q}} < \infty$.*

**Proposition 3.1** *Let $1 \leq p_n \leq \overline{p} < \infty$, $n \in \mathbb{N}$. Then the Nakano sequence space $\ell(\{p_n\})$ is saturated with $\ell_p$ iff $\lim_{n\to\infty} = p$.*

**Proof:** Let $\lim_{n\to\infty} p_n$, $Y$ be a infinite dimensional subspace of $\ell(\{p_n\})$ and $\{e_i\}_{i=1}^{\infty}$ is the unit vector basis of $\ell(\{p_n\})$ i.e. $e_i = \delta_{ij}$, $j = 1, 2, \ldots$.

Obviously the norm one vectors $u_i = p_i^{1/p_i} e_i$, $i = 1, 2, \ldots$ form an unconditional basis of $\ell(\{p_n\})$. According to [9] there is a subspace $Y_1$ of $Y$, which is isomorphic to a subspace generated by a block basis $\{v_k\}_{k=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$. Let $Y_1 = \overline{\operatorname{span}\{v_k\}_{k=1}^{\infty}}$, $v_k = \sum_{i \in A_k} a_i u_i$, $\|v_k\| = 1$, $k = 1, 2, \ldots$, $\{A_k\}_{k=1}^{\infty}$ is a family of finite pairwise disjoint subsets of $\mathbb{N}$.

Consider the functions $b_k(x) = \sum_{i \in A_k} |a_i x|^{p_i}$, $x \in [0, 1]$, $k = 1, 2, \ldots$. Obviously $b_k'(x) = \sum_{i \in A_k} p_i |a_i x|^{p_i - 1} |a_i| \leq \sum_{i \in A_k} p_i |a_i|^{p_i} \leq \overline{p}$, for every $k = 1, 2, \ldots$. Thus $\{b_k(x)\}$ is a family of uniformly bounded and equicontinuous functions and, therefore, there exists a continuous function $M(x)$ and a subsequence $\{b_{k_i}\}$ s.t. $|M(x) - b_{k_i}(x)| \leq 1/2^i$ for all $i \in \mathbb{N}$ and every $x \in [0, 1]$. It is easily seen that $M$ is an Orlicz function and $\ell_M \cong \overline{\operatorname{span}\{v_{k_n}\}_{k=1}^{\infty}} \subset Y_1$.

Let $q < p < r$. Then for sufficiently large $n$ and an arbitrary $u, v \in [0, 1]$ we can write

$$u^r b_{k_n}(v) \leq u^{\max\{p_i : i \in A_{k_n}\}} b_{k_n}(v) \leq b_{k_n}(uv) \leq u^{\min\{p_i : i \in A_{k_n}\}} b_{k_n}(v) \leq u^q b_{k_n}(v).$$

Thus, $u^r M(v) \leq M(uv) \leq u^q M(v)$ and, therefore, $\alpha_M = \beta_M = p$, because the numbers $q$ and $r$ were arbitrary chosen. According to ([5], p. 143) there is an isomorphic copy of $\ell_p$ in $\ell_M \subset Y_1$.

Let now $\ell(\{p_n\})$ is saturated with $\ell_p$. Then there is an isomorphic copy of $\ell_p$ in $\ell(\{p_n\})$ and by [8] follows that $p$ is an accumulation point for $\{p_n\}$. Suppose that $\lim_{n\to\infty} p_n \neq p$. Then $\lim_{k\to\infty} p_{n_k} = q \neq p$ for some subsequence $\{p_{n_k}\}$ which implies $\ell(\{p_n\})$ contains an isomorphic copy of $\ell_q$. Thus there is a subspace isomorphic to $\ell_p$ in $\ell_q$, $p \neq q$ which is a contradiction. $\square$

**Corollary 3.1** *Let $1 \le p_n$, $n \in \mathbb{N}$, $\lim_{n\to\infty} p_n = q$, $q$ is not an even number. Then there is no $E(q)+1$–times differentiable bump function in $\ell(\{p_n\})$.*

**Proof:** The Nakano sequence space $\ell(\{p_n\})$ is saturated with $\ell_q$, but the upper estimate for the Fréchet differentiability of bumps in $\ell_q$ is $E(q)$ if $q$ is not an even number [1]. $\square$

In particular, there is no $E(\underline{p})+1$–times differentiable bump in $\ell(\{p_n\})$, $\liminf_{n\to\infty} p_n = \underline{p}$ and according to Corollary 2.1 this estimate is exact.

**Corollary 3.2** *Let $p$ is an even number, $1 < p \le p_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} p_n = p$. Then there is $p$–times continuously differentiable bump function in $\ell(\{p_n\})$ iff $\ell(\{p_n\} \cong \ell_p$.*

**Proof:** Suppose that there is a bump function $b(x) \in C^p(X)$. The condition $\lim_{n\to\infty} p_n = p$ and Lemma 3.2 gives us that $\ell(\{p_n\})$ is saturated with spaces isomorphic to $\ell_p$. Thus $\ell(\{p_n\})$ is saturated with spaces with cotype $p$. According to [3] it follows that $\ell(\{p_n\})$ has cotype $p$. Now Lemma 3.1 and Lemma 3.2 imply that $\ell(\{p_n\})$ isomorphic to $\ell_p$.

If $\ell(\{p_n\})$ is isomorphic to $\ell_p$ it is well known that the canonical norm is infinitely many times Fréchet differentiable. $\square$

Using an idea that goes back to [1] one can easily prove the following:

**Lemma 3.3** *Let $X$ be a Banach space with $p$–convex unconditional basis ($\left\|\sum_{i=1}^{n} x_i\right\|^p \le \sum_{i=1}^{n} \|x_i\|^p$ for any finite system $\{x_i\}_{i=1}^n$ of disjoint vectors in $X$). Then $\lim_{n\to\infty} \|P\|_n = 0$ for each homogeneous polynomial $P$,* deg *$P < p$, where $\|P\|_n = \sup\left\{|P(x)| : x = \sum_{i=k}^{\infty} x_i, k > n, \|x\| \le 1\right\}$.*

Finally we solve the general case $\lim_{n\to\infty} p_n = 2k$, $k = 1, 2$ by proving the following:

**Proposition 3.2** *Let $p = 2k$, $k = 1, 2$ and $\lim_{n\to\infty} p_n = p$ and $p_n \le p$. Then there is a $p$–times differentiable bump in $\ell(\{p_n\})$ iff $\ell(\{p_n\}) \cong \ell_p$.*

**Proof:** The "if" part is trivial. Let $b$ is $p = 2k$–times Fréchet differentiable bump in $X = \ell_\psi$, $\psi = \{t^{p_n}\}_{n=1}^{\infty}$, $\ell_\psi \cong \ell(\{p_n\})$. WLOG we my suppose $2k-1 < q \le p_n$, $b(x) \equiv 0$ for $\|x\| \ge 1$ and $b(0) > 0$. For $x = \{x_j\}_{j=1}^{\infty} \in X$ denote $\widetilde{M}(x) = \sum_{j=1}^{\infty} |x_j|^{p_j}$. Consider $\delta(x) = b^{-2}(x)$, $b(x) \ne 0$ and

$$\varphi(x) = \begin{cases} \delta(x) - \widetilde{M}(x) + 2, & b(x) \ne 0, \\ +\infty, & b(x) = 0. \end{cases}$$

Obviously $\varphi$ is *lsc*, bounded bellow and $\varphi(h) \geq \|h\|$, $h \in X$. According to Stegall Variational Principle (see e.g. [4]) there exist $x^* \in X^*$ and $x \in X$, $\varphi(x) < \infty$, such that for every $h \in X$

$$\varphi(x+h) - \varphi(x) + x^*(h) \geq 0.$$

Obviously $b(x) \neq 0$ and we get immediately

$$\sum_{j=1}^{p} \frac{\delta^{(j)}(x; h^j)}{j!} + x^*(h) + o_x(\|h\|^p) \geq \widetilde{M}(x+h) - \widetilde{M}(x).$$

Summing the last inequality with its analogue for $-h$ we obtain for $p = 2k$, $k \in \mathbb{N}$

$$\text{(1)} \qquad \sum_{j=1}^{k} \frac{\delta^{(2j)}(x; h^{2j})}{(2j)!} + o_x(\|h\|^p) \geq \widetilde{M}(x+h) + \widetilde{M}(x-h) - 2\widetilde{M}(x).$$

Using $\widetilde{M}(x+h) + \widetilde{M}(x-h) - 2\widetilde{M}(x) \geq C \sum_{j=1}^{\infty} |h_j|^{p_j} \geq C\|h\|^p$, $\|h\| \leq 1$ and (1) we get

$$\text{(2)} \qquad \sum_{j=1}^{k} \delta^{(2j)}(x; h^{2j}) \geq C_1 \sum_{j=1}^{\infty} |h_j|^{p_j}.$$

for some $C_1 > 0$ and all $h$, $\|h\| \leq \lambda$, $\lambda$ sufficiently small. Suppose now that $X \not\cong \ell_p$. Then there exist sequence $\{h_j\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} |h_j|^{p_j} = \infty$ and $\sum_{j=1}^{\infty} |h_j|^p < \infty$.

Finding increasing sequence $\{k_n\}_{n=1}^{\infty}$ of naturals, such that for

$$z_n = \sum_{j \in D_n} h_j e_j, \quad D_n = \{k_n + 1, \ldots, k_{n+1}\}$$

we have $1 \leq \|z_n\| < 2$, $n \in \mathbb{N}$. Then $1 \leq \sum_{j \in D_n} |h_j|^{p_j} \leq 2^p$.

Let us apply (2) for $h = h_m e_m$, $m$ big enough. We get

$$\sum_{j=1}^{k} \frac{\delta^{(2i)}(x; e_m^{2i})}{(2i)!} h_m^{2i} \geq C_1 |h_m|^{p_m}$$

and after summation over $D_n$, $n$ big enough

$$\text{(3)} \qquad \sum_{i=1}^{k} \left\{ \sum_{j \in D_n} \frac{\delta^{(2i)}(x; e_j^{2i})}{(2i)!} h_j^{2i} \right\} \geq C_1.$$

We immediately get a contradiction for $p = 2$, taking into account in (3) that $|\delta^{(2)}(x; e_j^2)| \leq \|\delta^{(2)}(x)\|$ for each $j$.

As $|\delta^{(4)}(x;e_j^4)| \leq \|\delta^{(4)}(x)\|$ for each $j$, in order to prove the assertion for $p=4$ it is enough to prove that $\lim\limits_{n\to\infty} \sum\limits_{j\in D_n} \delta^{(2)}(x;e_j^2)h_j^2$ and to use (3) for $k=2$.

It is easy to observe that

$$\frac{1}{2^{|D_n|}} \sum_{\{\varepsilon_j=\pm 1, j\in D_n\}} \delta^{(2)}\left(\left(\sum_{j\in D_n} \varepsilon_j h_j e_j\right)^2\right) = \sum_{j\in D_n} \delta^{(2)}(x;e_j^2)h_j^2.$$

Then for some $\overline{z} = \sum\limits_{j\in D_n} \overline{\varepsilon}_j h_j e_j$, $\overline{\varepsilon}_j = \pm 1$ we have $\left|\sum\limits_{j\in D_n} \delta^{(2)}(x;e_j^2)h_j^2\right| \leq |\delta^{(2)}(x;\overline{z}_n^2)|$ and to finish the proof in this last case we only need to observe that $\ell_\psi$ satisfies upper 3–estimate and to use Lemma 3.3 to get $\lim\limits_{n\to\infty} |\delta^{(2)}(x;\overline{z}_n^2)$. $\square$

The case $p_n \leq 2k$, $k \in \mathbb{N}$, $k > 2$ is still open. Our conjecture is that if $\lim\limits_{n\to\infty} p_n = 2k$, $k\in\mathbb{N}$, $k>2$ then $\ell_{(\{p_n\})} \cong \ell_{2k}$ iff there is a $2k$–times differentiable bump in $\ell_{(\{p_n\})}$.

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