Cotype of Weighted Orlicz Sequence Spaces

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In [4] and [5], characterization has been found of the cotype of Banach lattices in terms of new modulus of convexity $\gamma_X(\varepsilon)$ called order modulus of convexity. Exact estimates for γ_X in most common Orlicz sequence and function spaces have been determined. Recently in [2] charactarization was given of the Musielak–Orlicz sequence spaces with power type and cotype. In [6] C. Ruiz proved a particular case of the above class of spaces, namely the weighted Orlicz sequence spaces $\ell_M(w)$ are all mutually isomorphic for the weight sequences $w = \{w_n\}_{n=1}^\infty$ verifying the conditions $\lim_{k\to\infty} w_{n_k} = 0$ and $\sum_{k=1}^\infty w_{n_k} = \infty$ for some subsequence .

Naturally the problem arises to find an sharp estimates for the order modulus of convexity $\gamma_{\ell_M(w)}$ and thus the best cotype in weighted Orlicz sequence spaces, verifying the above condition.

We recall that M is Orlicz function if M is even, convex, $M(0) = 0$, $M(t) > 0$ for $t > 0$. The Orlicz function $M(t)$ is said to have the property Δ_2 if there exists a constant c such that $M(2t) \le cM(t)$ for every t. For a positive measure space (Ω, Σ, μ) the Orlicz space $L_M(\mu)$ is defined as the set of all equivalence classes of μ–measurable scalar functions x on Ω such that for some $\lambda > 0$

$$\widetilde{M}(x/\lambda) = \int_\Omega M(x(t)/\lambda)d\mu(t) < \infty, \quad \|x\| = \inf\{\lambda > 0 : \widetilde{M}(x/\lambda) \le 1\}.$$

For $\Omega = \mathbb{N}$ and $w = \{w_j\}_{j=1}^\infty = \{\mu(j)\}_1^\infty$ we get the weighted Orlicz sequence spaces $\ell_M(w)$. In this case we have $x \in \ell_M(w)$ iff $\widetilde{M}(x/\lambda) = \sum_{i=1}^\infty w_i M(x_i/\lambda) < \infty$ for some $\lambda < \infty$.

It is well known that the space $\ell_M(w)$ endowed with the Luxemburg norm from above is a Banach space. The unit vectors form an unconditional basis in $\ell_M(w)$. When $w_j = 1$ for each j, we obtain the usual Orlicz sequence space denoted by ℓ_M.

By $w \in \Lambda$, we mean that there exists a subsequence $\{w_{j_k}\}_{k=1}^\infty$ of w such that

$$\lim_{k\to\infty} w_{j_k} = 0 \quad \text{and} \quad \sum_{k=1}^\infty w_{j_k} = \infty. \tag{1}$$

We shall call (see [4] and [5]) order modulus of convexity of the Banach lattice X the function

$$\gamma_X(\varepsilon) = \inf\left\{\left\|\sqrt{x^2+\varepsilon^2y^2}\right\| - 1 : \|x\| = \|y\| = 1\right\}, \ \varepsilon \ge 0$$

where by $\sqrt{x^2+\varepsilon^2y^2}$ in the sequence spaces we mean the sequence $\{\sqrt{x^2(j)+\varepsilon^2y^2(j)}\}_{j=1}^\infty$.

To every Orlicz function M the following number is associate (see [3], p. 143)

$$\beta_M = \inf\left\{q : \inf_{0<\lambda,t\le 1} \frac{M(\lambda t)}{M(\lambda)t^q} > 0\right\}$$

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We introduce the following function necessary for the estimation of $\gamma_{\ell_M(w)}(\varepsilon)$

$$F_{M,I}(\varepsilon) = \inf\left\{\frac{\varepsilon^2 M(uv)}{u^2 M(v)} : u \in [\varepsilon, 1], v \in I\right\}, \quad \varepsilon \in [0,1]$$

Let us note that, if M has not the property Δ_2 at 0 and ∞, then the function $F_{M,(0,\infty)}$ is identically zero on $(0,1)$.

If $f : [0,a) \to [0,\infty)$, we say (see [1]) that the Banach space X is of cotype f if a constants $c > 0$, $C > 0$ exist, such that for every finite system $\{x_i\}$ of elements in X if $\int_0^1 \|\sum r_i(t) x_i\| dt \le c$ then $\sum f(\|x_i\|) \le C$, $r_i(t) = \mathrm{sign}(\sin(2^i \pi t))$, $t \in [0,1]$ are the first n Rademacher functions.

Theorem 1 *Let $X = \ell_M(w)$ be a weighted Orlicz sequence space. Then the following estimate holds:*

$$\gamma_X(\varepsilon) \ge C_M F_{M,(0,\infty)}, \quad \varepsilon \in [0,1]. \tag{2}$$

If in addition $w \in \Lambda$ and $|\cdot|$ is an arbitrary equivalent norm in $\ell_M(w)$ then

$$\gamma_{(X,|\cdot|)}(\varepsilon) \le C_M F_{M,(0,\infty)}(\varepsilon), \quad \varepsilon \in [0,1] \tag{3}$$

Proof: The estimate (2) is obtained following the method from [5].

We prove here the second part of the theorem. If M does not satisfy the Δ_2 condition then $\gamma(\varepsilon) \equiv 0$, $F_{M,(0,\infty)}(\varepsilon) \equiv 0$ for $\varepsilon \in (0,1)$. Let now M satisfies the Δ_2 and let $|\cdot|$ be a norm in $\ell_M(w)$, equivalent to $\|\cdot\|$, i.e. there exists constant a such that $a^{-1}|x| \le \|x\| \le a|x|$, for every $x \in \ell_M(w)$. It is enough to prove the estimate (3) in a small interval $[0, \varepsilon_0]$. WLOG we may assume that $\{w_k\}_{k=1}^\infty$ is non–increasing, $\lim_{k\to\infty} w_k = 0$ and $\sum_{k=1}^\infty w_k = \infty$.

Now let $\varepsilon \in (0, 1/6a^2)$ and $u \in [\varepsilon, 1]$, $v \in (0,\infty)$. Put $d = 6a^2$. We consider two cases.

Let first $u < d^{-1}$. Choose numbers p_1 and q_1 with the property

$$1/M(2v) \le \sum_{j=p_1}^{q_1} w_j \le 1/M(v). \tag{4}$$

Let $z_1 = uM^{-1}\left(1/\sum_{j=p_1}^{q_1} w_j\right)$ and x_1 be the vector with finite support in $\ell_M(w)$ defined by $x_1 = \sum_{j=p_1}^{q_1} z_1 e_j$. The equality $\widetilde{M}(x_1/u) = \sum_{j=p_1}^{q_1} w_j M(z_1/u) = 1$ implies $\|x_1\| = u$.

Obviously

$$\widetilde{M}(dx_1) \sum_{j=p_1}^{q_1} w_j M(dz_1) = \sum_{j=p_1}^{q_1} w_j M\left(duM^{-1}\left(\frac{1}{\sum_{p_1}^{q_1} w_j}\right)\right) < \sum_{j=p_1}^{q_1} w_j M\left(M^{-1}\left(\frac{1}{\sum_{p_1+1}^{q_1} w_j}\right)\right) = 1.$$

So we can find an integer $n \in \mathbb{N}$ such that

$$1/2 \le n\widetilde{M}(dx_1) = n\sum_{j=p_1}^{q_1} w_j M(dz_1) < 1. \tag{5}$$

Let $\delta = \left(n^{-1} - \widetilde{M}(dx_1)\right)/M(dz_1)$. Now we produce a sequence of $n-1$ more pairs of integers $\{p_i, q_i\}_{i=2}^n$ such that $q_{i-1} < p_i \le q_i$ for $i = 2 \ldots n$ and

(6)
$$\sum_{j=p_1}^{q_1} w_j \le \sum_{j=p_i}^{q_i} w_j \le \sum_{j=p_1}^{q_1} w_j + \delta.$$

Let $\{x_i\}_{i=2}^n$ be the sequence of vectors with finite supports defined by $x_i = \sum_{j=p_i}^{q_i} z_1 e_j$. Using inequality (6) we obtain

$$\widetilde{M}(x_i/u) = \sum_{j=p_i}^{q_i} w_j M\left((u/u)M^{-1}\left(1/\sum_{j=p_1}^{q_1} w_j\right)\right) = \frac{\sum_{j=p_i}^{q_i} w_j}{\sum_{j=p_1}^{q_1} w_j} \ge \frac{\sum_{j=p_1}^{q_1} w_j}{\sum_{j=p_1}^{q_1} w_j} = 1,$$

which implies that $\|x_i\| \ge u$ for every $i = 2 \dots n$. Using again the inequality (6) we find that

$$\left|\widetilde{M}(dx_1) - \widetilde{M}(dx_i)\right| = M(dz_1)\left|\sum_{j=p_1}^{q_1} w_j - \sum_{j=p_i}^{q_i} w_j\right| \le n^{-1} - \widetilde{M}(dx_1), \quad i = 1, \dots, n.$$

(7)
$$\left|n\widetilde{M}(dx_1) - \widetilde{M}\left(\sum_{i=1}^n dx_i\right)\right| \le n\left(n^{-1} - \widetilde{M}(dx_1)\right) = 1 - n\widetilde{M}(dx_1)).$$

From (5) and (7) we obtain
$$\widetilde{M}\left(d\sum_{j=1}^n x_i\right) \le 1.$$

The last inequality shows that $|\sum_{i=1}^n \varepsilon_i 6ax_i| = \|\sum_{i=1}^n \varepsilon_i dx_i\| \le 1$, where $\varepsilon_i = \pm 1$.

Using Proposition 3.5 [5] we obtain that $\sum_{i=1}^n \gamma(\|x_i\|) \le \sum_{i=1}^n \gamma(a|x_i|) \le 1$. Therefore,

(8)
$$n\gamma(u) = \sum_{i=1}^n \gamma(u) \le \sum_{i=1}^n \gamma(\|x_i\|) \le 1.$$

From (4) we obtain $M(v) \le \dfrac{1}{\sum_{j=p_1}^{q_1} w_j} \le M(2v)$. Obviously from the inequalities

$$M^{-1}(M(v)) \le M^{-1}\left(\frac{1}{\sum_{j=p_1}^{q_1} w_j}\right) \le M^{-1}(M(2v))$$

follows that

(9)
$$uv \le z_1 \le 2uv.$$

As $\gamma(u)/u^2$ is equivalent to a non-decreasing function (see Lemma 3.11[5]) using consecutively the inequalities (8), (5), (4) and (9) we obtain

$$\begin{aligned}\gamma(\varepsilon) &\le c\gamma(u)\left(\frac{\varepsilon}{u}\right)^2 \le cn^{-1}\left(\frac{\varepsilon}{u}\right)^2 \le 2c\left(\frac{\varepsilon}{u}\right)^2 \sum_{j=p_1}^{q_1} w_j M(6z_1) \\ &\le c\left(\frac{\varepsilon}{u}\right)^2 \frac{M(6z_1)}{M(v)} \le c\left(\frac{\varepsilon}{u}\right)^2 \frac{M(12uv)}{M(v)} \le c(\Delta_2)\left(\frac{\varepsilon}{u}\right)^2 \frac{M(uv)}{M(v)},\end{aligned} \tag{10}$$

where $c(\Delta_2)$ is a constant depending only on M, related to the Δ_2–condition.

Now let $u \ge d$. Using the Δ_2–condition it is easily seen that

$$\left(\frac{\varepsilon}{u}\right)^2 \frac{M(uv)}{M(v)} \ge \varepsilon^2 \frac{M(dv)}{M(v)} \ge c(\Delta_2)\varepsilon^2 \ge \ 2c\gamma(\varepsilon). \tag{11}$$

As the inequalities (10) and (11) are fulfilled for arbitrary $u \in [\varepsilon, 1]$ and $v \in (0,\infty)$ it follows the existence of a constant C_M, such that $\gamma(\varepsilon) \le C_M F_{M,(0,\infty)}(\varepsilon)$. □

Using the characterization of the cotype in Banach lattices

Theorem 2 *(Theorem 3.13 [5]) A Banach lattice X is of cotype f iff there exists in X an equivalent norm $|\cdot|$ and constants $c_1, c_2 > 0$, so that for every $\varepsilon \in [0, c_2]$ is fulfilled:*

$$\gamma_{(X,|\cdot|)}(\varepsilon) \ge c_1 f(\varepsilon),$$

and that (see Theorem 4.9 in [5]) $F_{M,(0,\infty)}(\varepsilon) \ge c\varepsilon^p$ iff the function $M(t^{1/p})$ is quasiconcave, i.e.

$$M\left(\left(\sum_{i=1}^n \frac{t_i}{n}\right)^{1/p}\right) \ge c\frac{\sum_{i=1}^n M\left(t_i^{1/p}\right)}{n} \quad \text{for} \quad t_i \in (0,\infty),$$

we obtain the following:

Theorem 3 *Let $\ell_M(w)$ be a weighted Orlicz sequence space. Then*

a) $\ell_M(w)$ is of cotype $F_{M,(0,\infty)}(\varepsilon)$. If in addition $w \in \Lambda$ then $F_{M,(0,\infty)}(\varepsilon)$ is the exact cotype of $\ell_M(w)$.

b) $\ell_M(w)$ is of cotype p, $2 \le p < \infty$ iff the function $M(t^{1/p})$ is quasiconcave in $[0,\infty)$.

References

[1] T. Figiel. On the moduli of convexity and smoothness. *Studia Math.*, **56**, (1976), 121-155.

[2] E. Katirtzoglou. Type and Cotype of Musielak–Orlicz sequence Spaces *J. Math. Anal. Appl.*, **226** (1998), 431-455.

[3] J. Lindenstrauss, L. Tzafriri. "Classical Banach Spaces. I. Sequence Spaces" Springer–Verlag, New York, 1979.

[4] R. Maleev, S. Troyanski Order moduli of convexity and smoothness *Funct. Anal. Appl.*, **17** (1983), 231–233.

[5] R. Maleev, S. Troyanski. On Cotypes of Banach Lattices. *Constructive Function theory'81*, Sofia (1983), 429-441.

[6] C. Ruiz. On the Structure of Rosenthal's Space X_φ in Orlicz Function Spaces. *Rocky Mount. Math. J.* **27** (1997), 1223–1238.

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