LH^{ω}–Smooth Bump Functions in Weighted Orlicz Sequence Spaces

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Abstract An exact estimate is given for the modulus of smoothness in weighted Orlicz sequence spaces and the best order of LH^{ω}–smoothness of bump functions is found for $\alpha_M \leq 2$.

Keywords: weighted Orlicz sequence space, smooth bump function

1 Introduction

For many problems in the Geometry of Banach spaces and non linear analysis in Banach spaces, the existence of bump function with prescribed order of smoothness or with derivatives sharing properties of Holder's type is of essential importance.

As a Frechet smooth norm immediately produces a bump function of the same smoothness, all negative results about bump functions are negative results about the smoothness in the class of all equivalent norms.

The question of finding an upper bound of the order of Frechet differentiability of bump functions in arbitrary Orlicz space is solved in [10]. Recently in [11] Ruiz has proved that for a given Orlicz function M all weighted Orlicz sequence spaces $\ell_M(w)$, generated by a weight sequence $w = \{w_n\}_{n=1}^{\infty}$ verifying the condition

$$(1) \qquad \lim_{k\to\infty} w_{j_k} = 0, \quad \sum_{k=1}^{\infty} w_{j_k} = \infty,$$

for some subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, are mutually isomorphic. This result raises the question whether the best possible ω_1–Holder properties of the first derivatives of bump functions in $\ell_M(w)$ depend on the sequence $w = \{w_n\}_{n=1}^{\infty}$ verifying (1)

For the proof of the main result we shall need an estimate from below of the modulus of smoothnes in weighted Orlicz sequence space $\ell_M(w)$. Maleev and Troyanski [9] have found an upper estimate for the modulus of smoothness of an arbitrary Orlicz space. Figel in [3] has shown that this estimate is exact up to equivalent renorming in Orlicz spaces. Using the method of Figel, we will show in Section 4, Lemma 1, that the estimates found in [9] are exact up to equivalent renorming also weighted Orlicz sequence space.

2 Preliminary results

We denote by X a Banach space, X^* its dual one, S_X the unit sphere in X, $\mathbb{N}$ the naturals and $\mathbb{R}$ the reals. Everywhere differentiability is understood as Frechet differentiability.

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An Orlicz function M is an even, continuous, convex and monotone function in $[0,\infty)$ with $M(0)=0$, $M(t)>0$ for any $t\neq 0$. The Orlicz function M is said to have the property Δ_2 if there exists a constant C such that $M(2t)\leq CM(t)$ for every $t\in[0,\infty)$.

To every Orlicz function M the following numbers are associated

$$\alpha_M^0 = \sup\left\{p : \sup_{0<\lambda,t\leq 1} \frac{M(\lambda t)}{M(\lambda)t^p} < \infty\right\},$$

$$\alpha_M^\infty = \sup\left\{p : \sup_{1\leq\lambda,t<\infty} \frac{M(\lambda)t^p}{M(\lambda t)} < \infty\right\},$$

$\alpha_M = \min\{\alpha_M^0, \alpha_M^\infty\}$. (see e.g.[5], p. 143 and [6] p. 382)

Let (S,Σ,μ) be a positive measure space. The Orlicz space $L_M(\mu)$ is defined as the set of all equivalence classes of μ–measurable scalar functions x on S such that

$$\widetilde{M}(x/\lambda) = \textstyle\int_\Omega M(x(t)/\lambda)d\mu(t) < \infty$$

for some $\lambda>0$, equipped with the Luxemburg's norm

$$\|x\| = \inf\left\{\lambda>0 : \widetilde{M}(x/\lambda)\leq 1\right\}. \tag{2}$$

For $S=\mathbb{N}$ and $w=\{w_j\}_{j=1}^\infty = \{\mu(j)\}_{j=1}^\infty$ we get the weighted Orlicz sequence spaces $\ell_M(w)$. In this case we have $x=\{x_j\}_{j=1}^\infty\in\ell_M(w)$ iff there exists $\lambda>0$:

$$\widetilde{M}(x/\lambda) = \sum_{j=1}^\infty w_j M(x_j/\lambda) < \infty.$$

Clearly, the unit vector sequence is an unconditional basis in $\ell_M(w)$. When $w_j=1$ for each $j\in\mathbb{N}$, we obtain the usual Orlicz sequence space and denote it by ℓ_M instead of $\ell_M(w)$.

Let $w=\{w_j\}_{j=1}^\infty$ be a sequence and $w_j>0$ for every $j\in\mathbb{N}$. By $w\in\Lambda$, we mean that there exists a subsequence $\{w_{j_k}\}_{k=1}^\infty$, verifying conditions (1)

We call modulus of smoothness of X the function:

$$\rho_X(\tau) = \frac{1}{2}\sup\left\{\|x+\tau y\| + \|x-\tau y\| - 2 : x,y\in S_X\right\},\quad \tau>0.$$

We introduce the following function necessary for the estimation of $\rho_{(\ell_M(w),|\cdot|)}(\tau)$ with respect to an appropriate equivalent norm $|\cdot|$ in $\ell_M(w)$

$$G_M(\tau) = \tau^2\sup\left\{\frac{M(uv)}{u^2M(v)} : u\in[\tau,1], v>0\right\},\quad \tau\in(0,1]$$

The function $f : X\to\mathbb{R}$ is said to be differentiable at $x\in X$ if there exists $z_x^*\in X^*$, such that

$$f(x+ty) = f(x) + tz_x^*(y) + r(x,y,t), \tag{3}$$

where $\lim_{t\to 0} t^{-1}\sup\{r(x,y,t) : y \in S_X\} = 0$. The functional z_x^* is called derivative of f at x and is denoted by $f'(x)$.

In the applications often are considered functions, which are not only differentiable but share properties of Hölder's type.

By Ω we denote the class of functions $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t) = o(t)$ and $\omega_1(t) = \omega(t)/t$ is nondecreasing function, satisfying the condition $\omega_1(\lambda t) \leq \lambda\omega_1(t)$ for every $\lambda \geq 1$.

We say that $f : X \to \mathbb{R}$ is locally H^ω–smooth in the open subset $V \subset X$ if f is continuously differentiable in V and for every $x \in V$ there exist $\delta = \delta(x) > 0$ and $A = A(x) > 0$ such that

$$\|f'(y) - f'(z)\| \leq A\omega_1(\|y - z\|) = A\frac{\omega(\|y - z\|)}{\|y - z\|} \tag{4}$$

for every $y, z \in B(x;\delta) \subset V$ (see [1]).

If there exists $A > 0$, such that (4) is fulfilled for arbitrary $y, z \in V$ the function f is called H^ω–smooth in V. The class of all H^ω–smooth (locally H^ω–smooth) functions in V will be denoted by $H^\omega(V)$ $(LH^\omega(V))$, respectively.

We say that $b : X \to \mathbb{R}$ is a bump function if supp $b = \overline{\{x \in X; b(x) \neq 0\}}^{\|\cdot\|}$ is a bounded non empty set.

It is easy to observe that if there exist $H^\omega(X)$ $(LH^\omega(X))$–smooth equivalent norm, then there exists a $H^\omega(X)$ $(LH^\omega(X))$–smooth bump (see e.g. [2] p. 9). The converse is not true. Haydon [4] gives an example of space with C^∞–smooth bump, which has not even a Gâteaux differentiable equivalent norm.

3 Main result

Theorem 1 *Let $X = \ell_M(w)$, where M is an Orlicz function, satisfying the Δ_2–condition at 0 and at ∞, $\alpha_M \in (1, 2]$, $w \in \Lambda$ and $\omega \in \Omega_1$. If b is LH^ω–smooth bump function in $\ell_M(w)$ then*

$$G_M(\tau) = O(\omega(\tau)).$$

4 Modulus of smoothness of weighted Orlicz spaces

In the proof of the next Lemma 1 we shall use the following result

Proposition 1 *([3], [6]). There exists a positive absolute constant L such that $\rho_X(\sigma)/\sigma^2 \leq L\rho_X(\tau)/\tau^2$, whenever $0 < \tau < \sigma$.*

Lemma 1 *Let $X = \ell_M(w)$, where M is an Orlicz function, satisfying the Δ_2–condition and $w \in \Lambda$. Then for every equivalent norm $|\cdot|$ on X, there exists a constant $K = K_{|\cdot|} > 0$, such that*

$$\rho_{(X,|\cdot|)}(\tau) \geq KG_M(\tau), \quad \tau \in (0, 1].$$

Proof: We can assume WLOG that $|x| \leq \|x\| \leq b|x|$ for every $x \in X$, where $\|\cdot\|$ is the Luxemburg norm (2). As the norm $|\cdot|$ is fixed we can denote $\rho(\tau) = \rho_{(X,|\cdot|)}(\tau)$ and we shall denote the subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ fulfilling (1) again by $\{w_j\}_{j=1}^{\infty}$ just for simplicity.

Observe that from the equivalence of the norms it follows that $\sum_{i=1}^{n} \rho(|x_i|) \leq 1$, provided $\sum_{i=1}^{n} \rho(\|x_i\|) \leq 1$. Hence by Lindenstrauss' Theorem (in the Figiel's form [3]) there exist signs $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$ so that $|\sum_{i=1}^{n} \varepsilon_i x_i| \leq 1 + \sqrt{3}$, which gives us that

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leq (1+\sqrt{3})b = d, \tag{5}$$

whenever $\sum_{i=1}^{n} \rho(\|x_i\|) \leq 1$.

For every given $\tau, u \in [\tau, 1]$, $v \in (0, \infty)$ we put $n = [1/\rho(u)]$, $c = uv$, where by $[a]$ we denote the largest integer not greater than a.

For every v we can choose a sequence of integers $\{m_k\}_{k=1}^{\infty}$: $\frac{1}{2M(v)} < \sum_{j=m_k+1}^{m_{k+1}} w_j < \frac{1}{M(v)}$, because $w \in \Lambda$. Let $x_k \in X$, $k = 1, \ldots, n$ be disjointly supported vectors such that

$$x_k = c \sum_{j=m_k+1}^{m_{k+1}} e_j,$$

where $\{e_j\}_{j=1}^{\infty}$ is the unit vector basis in X. Obviously

$$1 = \sum_{j=m_k+1}^{m_{k+1}} w_j M(c/\|x_k\|) = M(c/\|x_k\|) \sum_{j=m_k+1}^{m_{k+1}} w_j < \frac{M(c/\|x_k\|)}{M(v)}$$

So we obtain that $\|x_k\| \leq u$, which yields the inequalities $\sum_{i=1}^{n} \rho(\|x_i\|) \leq n\rho(u) \leq 1$.

Using (5), we obtain immediately the inequalities

$$1 \geq \sum_{k=1}^{n} \sum_{j=m_k+1}^{m_{k+1}} w_j M(c/d) = M(c/d) \sum_{k=1}^{n} \sum_{j=m_k+1}^{m_{k+1}} w_j \geq M(c/d) \frac{n}{2M(v)}.$$

Hence

$$\frac{M(c/d)}{M(v)} \leq \frac{2}{n} \leq 2\frac{2}{n+1} \leq 4\rho(u).$$

Since $c = uv$, then there exists a constant α, depending only on d and the Δ_2–condition, such that $M(uv) \leq \alpha M(c/d)$. Finally, we obtain

$$\frac{M(uv)}{M(v)} \leq \alpha \frac{M(c/d)}{M(v)} \leq 4\alpha\rho(u). \tag{6}$$

To finish the proof we need only to apply (6) and Proposition 1. Indeed,

$$G_M(\tau) = \tau^2 \sup_{u \in [\tau,1], v>0} \frac{M(uv)}{u^2 M(v)} \leq \tau^2 \sup_{u \in [\tau,1], v>0} 4\alpha \frac{\rho(u)}{u^2} \leq \tau^2 4\alpha L \frac{\rho(\tau)}{\tau^2}.$$

$\square$

Combining the result in [9] with Lemma 1 we find that the estimate of the modulus of smoothness in weighted Orlicz sequence spaces is exact up to an equivalent renorming.

5 Proof of the Main Result

In the proof of Theorem 1 we shall need a variant of known theorems (see e.g. [2], p. 199). As the proofs are literally the same we shall omit them.

Theorem 2 *([2],5.3.1) Assume that a Banach space $X \not\supset c_0$. Suppose that X admits a bump function $b(x) \in LH^\omega(X)$. Then X admits a bump function $f(x) \in H^\omega(X)$.*

Theorem 3 *([2],5.3.2) Assume that a Banach space admits admits a bump function $b(x) \in H^\omega(X)$. Then X admits an equivalent norm $|\cdot| \in H^\omega(S_X)$.*

Proof of Theorem 1: Let f be a LH^ω–smooth bump in $X = \ell_M(w)$. Since there is no isomorphic copy of c_0 in X, then according to Theorem 2 there is a H^ω–smooth bump function in X. According to Theorem 3, there is an equivalent $H^\omega(S_X)$–smooth norm $|||\cdot|||$ such that

(7) $$\rho_{X,|||\cdot|||}(t) \le K\omega(t), \quad t \ge 0, \quad K > 0$$

On the other hand, we have just proved that the best order of the modulus of smoothness of any equivalent renorming of X is $G_M(t)$, i.e.

(8) $$\rho_{X,|||\cdot|||}(t) \ge cG_M(t), \quad c = c_{|||\cdot|||} > 0$$

for every equivalent norm $|||\cdot|||$ in X. Combining (7) and (8) we obtain

$$G_M(t) \le \frac{K}{c}\omega(t)$$

□

Remark: If $\alpha_M = 2$ and M, satisfying the condition

$$\sup\left\{\frac{M(uv)}{u^{\alpha_M}M(v)} : u, v \in (0,\infty)\right\},$$

is solved in [8]

Remark: If $M \sim t^2$, then there exists an equivalent, infinitly many times Frechet differentiable norm, and it is seen right away that $G_M(\tau) = \tau^2$, so there is nothing to be proved.

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