Gâteaux Differentiable Norms in Musielak–Orlicz Spaces [1]

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ABSTRACT Equivalent $[\alpha(\Phi)]$–times uniformly Gâteaux differentiable norms are constructed for large classes of Musielak–Orlicz spaces $L_\Phi(\Omega, \Sigma, \mu)$. Equivalent $[p]$–times uniformly Gâteaux differentiable norms are constructed in Nakano sequence spaces $\ell_{\{p_n\}}$, where $1 \leq p = \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n < \infty$ and the set $A = \{n \in \mathbb{N} : p_n < p\}$ is finite.

1 Introduction.

The existance of smooth bump functions on a Banach spaces is important for many problems of nonlinear analysis [3], [4]. It is well known that in a Banch space an equivalent norm of some order of smoothness generates a bump with the same order of smoothness. Thus all positive results on the existence of smooth equivalent norms are transferred directly for bumps.

The problem of best order of Fréchet differentiability of bump functions for L_p–spaces is solved in [1], [12] and for Orlicz spaces in [9], [10]. An excelent overview of the depelopment of the smoothness problems in Banach spaces may be found in [5].

Estimates for the order of Fréchet differentiability of norms in Musielak–Orlicz sequence spaces and of bump functions in Nakano sequence spaces habe been found in [11]. Troyanski [13] found equivalent p–times Gâteaux differentiable equivalent norms in L_p over σ–finite space, p odd, thus showing that the best order of Gâteaux differentiability in the class of all equivalent norms in this case is better that the best order of Fréchet differentiability. The same phenomenon was established in [8] for Orlicz sequence and function spaces.

Our aim is to constract equivalent norms, that are uniformly Gâteaux differentiable in some Musielak–Orlicz and Nakano spaces and to find cases when the order of Gâteaux differentiability of the norm is higher that the best order of Fréchet smoothness of the space.

2 Preliminaries.

In what follows X and Y are Banach spaces, S_X and B_X the unit sphere and the unit ball of X respectively. By $\mathbb{N}$ we denote the naturals and by $\mathbb{R}$ the reals. The space of all continuous symmetric j–linear forms

$$T : \underbrace{X \times X \times \cdots \times X}_{j-\text{times}} \to Y$$

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equiped with the norm

$$\|T\|_1 = \sup\{\|T(x_1, \dots, x_j)\| : x_i \in B_X, 1 \le i \le j\}$$

is denoted by $B^j(X,Y)$. We write $T(\underbrace{x, \dots, x}_{j-\text{times}}) = T(x^j)$.

Definition 2.1 *The map $f : X \to Y$ is said to be Gâteaux differentiable at $x \in X$ if for each $h \in X$*

$$f^{'}(x;h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

exists and is a linear continuous function in h, i.e. $f^{'} \in B(X,Y)$. The higher order Gâteaux differentiability is defined inductively. Suppose the $(k-1)$–th derivative $f^{(k-1)}$ of f is defined in a neighbourhood $U(x)$ of x, $f^{(k-1)}(y) \in B^{k-1}(X,Y)$ for every $y \in U(x)$. Then f is called k–times Gâteaux differentiable at x if $f^{(k-1)} : U(x) \to B^{k-1}(X,Y)$ is Gâteaux differentiable at x, i.e. if there exists $f^{(k)}(x) \in B^k(X,Y)$ such that for each $h \in X$

$$\lim_{t\to 0} \frac{f^{(k-1)}(x+th : \cdot) - f^{(k-1)}(x : \cdot)}{t} = f^{(k)}(x : \cdot h), \tag{1}$$

where the limit is understood with respect to the norm in $B^{(k-1)}(X,Y)$.

The class of all k–times Gâteaux differentiable maps at any $x \in A \subset X$ is denoted by $G^k(A)$. If $f \in G^k(S_X)$ and the limit in (1) is uniform on x for every fixed h then we say that f is k–times uniformly Gâteaux differentiable on S_X and we write $f \in UG^k(S_X)$.

We mention that if the limit s above are uniform for $h \in S_X$ the map is said Fréchet (k–times Fréchet) differentiable in x. A Banach space which possesses k–times Gâteaux (Fréchet) differentiable bump function (real valued function with bounded support) is called $G^k(F^k)$–smooth.

Throughout the paper (Ω, Σ, μ) is a measure space.

We recall that M is called an Orlicz function, provided M is even, convex, continuous nondecreasing in $[0,\infty)$ function with $M(0) = 0$, $M(t) > 0$ for any $t \ne 0$.

Definition 2.2 *A two variable real valued function $\Phi(u,s) : [0,\infty) \times \Omega \to [0,\infty)$ is called a Musielak–Orlicz function with a parameter or a Musielak–Orlicz function or MO function if for a.a. $s \in \Omega$, $u \to \Phi(u,s)$ is an Orlicz function and for all $u \in [0,\infty)$, $s \to \Phi(u,s)$ is Σ–measurable.*

Definition 2.3 *The Musielak–Orlicz space $L_\Phi(\Omega, \Sigma, \mu)$ is the space of all classes f of equivalent μ–mesurable functions over the measure space (Ω, Σ, μ) such that:*

$$\widetilde{\Phi}\left(\frac{f}{\lambda}\right) = \int_\Omega \Phi\left(\frac{f(s)}{\lambda}, s\right) d\mu(s) < \infty$$

for some $\lambda > 0$.

The space $L_\Phi(\Omega, \Sigma, \mu)$ is a Banach space if endowed with the Luxemburg's norm:

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \widetilde{\Phi}\left(\frac{f}{\lambda}\right) = 1\right\}.$$

The most common examples of Musielak–Orlicz spaces are the sequence spaces $\ell_{\{\Phi_n\}}$, the function spaces $L_\Phi(0,1)$ and $L_\Phi(0,\infty)$ that correspond to the cases: Ω countable union of atoms of equal mass, $\Omega = [0,1]$ and $\Omega = (0,\infty)$, μ the usual Lebesgue measure.

Definition 2.4 *We say that the Musielak–Orlicz function $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies the Δ^{*p} condition if there exist a positive constant k and a nonnegative integrable over Ω function h, such that:*

$$\Phi(\lambda t, s) \geq k\lambda^p(\Phi(t,s) - h(s)) \tag{2}$$

for any $t \geq 0$ and $\lambda \geq 1$.

If $h \equiv 0$ then we say that Φ satisfies the uniform Δ^{*p} condition.

Definition 2.5 *We say that the Musielak–Orlicz function $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies the Δ^q condition if there exist a positive constant K and a nonnegative integrable over Ω function h, such that:*

$$\Phi(\lambda t, s) \leq K\lambda^q(\Phi(t,s) + H(s)) \tag{3}$$

for any $t \geq 0$ and $\lambda \geq 1$.

If $H \equiv 0$ then we say that Φ satisfies the uniform Δ^q condition.

Definition 2.6 *Given a Musielak–Orlicz function Φ, lower and upper indices $\alpha(\Phi)$, $\beta(\Phi)$ respectively are defined as follows:*

$$\alpha(\Phi) = \sup\{p : \Phi \in \Delta^{*p}\}, \quad \beta(\Phi) = \inf\{q : \Phi \in \Delta^q\}.$$

Definition 2.7 *Let $\{p_n\}_{n=1}^\infty$, $p_n \geq 1$. The space of all sequences $\{x_n\}$ such that*

$$\widetilde{N}(x/\lambda) = \sum_{n=1}^\infty \left|\frac{x_n}{\lambda}\right|^{p_n} < \infty$$

for some $\lambda > 0$ is called a Nakano sequence space and denoted by $\ell_{\{p_n\}}$.

We consider the usual Luxemburg's norm.

3 Main results

Theorem 1 *Let Φ be a Musielak–Orlicz function with $1 \le \alpha(\Phi) \le \beta(\Phi) < \infty$ and let for some $k \in [1, \alpha(\Phi)]$:*
i) there exist non–negative integrable over Ω function h and positive constant c_0 such that:

$$\Phi(uv, s) \le c_0 u^k (\Phi(v, s) + h(s))$$

for all $u \in [0, 1]$, $v \in \mathbb{R}^+$ and a.a. $s \in \Omega$;
ii) $\lim\limits_{u\to 0} \dfrac{\Phi(u, s)}{u^k} = 0$ *for a.a. $s \in \Omega$.*
Then for any measure space (Ω, Σ, μ) with a positive measure there is an equivalent UG^k–smooth norm in $L_\Phi(\Omega, \Sigma, \mu)$.

Theorem 2 *Let $\{p_n\}_{n=1}^{\infty}$ be such that $1 \le p = \liminf\limits_{n\to\infty} p_n \le \limsup\limits_{n\to\infty} p_n < \infty$ and the set $A = \{n \in \mathbb{N} : p_n < p\}$ is finite. Then there exist an equivalent UG^p–smooth norm in $\ell_{\{p_n\}}$.*

4 Auxiliary results

4.1 Uniform Gâteaux differentiability of Musielak–Orlicz spaces

Following [8] for any $c \ge 1$ we denote by $F(c)$ the class of all non–decreasing continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f \not\equiv 0$, such that:
$(\star)$ $f(0) = 0$;
$(\star\star)$ $f(b) - f(a) \le c(b - a)\frac{f(b)}{b}$ for any $0 \le a \le b$, $b \ne 0$.
Let $f(t, s) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$. Denote $I(f) = I(f)(u, s) = \int_0^t f(u, s)du$ and inductively $I^n(f) = I^n(f)(u, s) = I(I^{n-1}(f)(u, s))$.

Theorem 3 *Let $f(t, s) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ and $f \in F(c)$ for a fixed $c \ge 1$ and almost all $s \in \Omega$. Then for every measure space (Ω, Σ, μ) with positive measure μ the Musielak–Oricz function space $L_\Phi(\Omega, \Sigma, \mu)$, $\Phi = I^k(f)$ admits equivalent UG^k–smooth norm.*

We omit here the proof because it can be obtained by carefully following step by step the proof from [8] of the analogous result for Orlicz spaces.

4.2 Musielak–Oricz functions satisfying simultaneously uniform Δ^{*p} and Δ^q conditions

Definition 4.1 *We say that Φ_1 is equivalent to Φ_2 and we denote it by $\Phi_1 \sim \Phi_2$ if there are constants C_i, K_i, $i = 1, 2$ and non–negative integrable over Ω functions h_i, $i = 1, 2$ satisfying*

$$C_1\Phi_1(K_1 u, s) \le \Phi_2(u, s) + h_1(s), \tag{4}$$

$$C_2\Phi_2(K_2u,s) \le \Phi_1(u,s) + h_2(s) \tag{5}$$

for all $u \ge 0$ and a.a. $s \in \Omega$.

It is well known that for equivalent Musielak–Orlicz functions Φ_1 and Φ_2 the Musielak–Orlicz spaces L_{Φ_1} and L_{Φ_2} are isomorphic (e.g. [6])

It is well known (e.g. [6],[7]) that if a Musielak–Orlicz function Φ satisfies the Δ^{*p} (Δ^q) then there exists an eqivalent Musielak–Oricz function Ψ which satisfies the uniform Δ^{*p} (Δ^q) condition. For the proof of the main result we will need the following stronger result which could be of interest by itself.

Theorem 4 *Let Φ satisfies the Δ^{*p} and Δ^q conditions for some $1 \le p \le q < \infty$. Then there exists $\overline{\Phi} \sim \Phi$ and positive constants k_1, K_1, such that*

$$k_1\lambda^p\overline{\Phi}(t,s) \le \overline{\Phi}(\lambda t,s) \le K_1\lambda^{p+q+1}\overline{\Phi}(t,s) \tag{6}$$

for any $\lambda \ge 1$, $t \ge 0$, $s \in \Omega$, i.e. $\overline{\Phi}$ satisfies the uniform Δ^{*p} and Δ^{p+q+1} conditions simultaniously.

Proof: Suppose that Φ satisfies (2) and (3).

Denote $b_1(s) = \Phi^{-1}(2h(s))$, $B_1(s) = \Phi^{-1}(H_1(s))$, $b(s) = \max\{b_1(s), B_1(s)\}$. Obviously $b_1, B_1, b \in L_\Phi$ are non–negative functions. Following [6] one can easily prove that there exist positive constants k_2 and K_2 such that

$$k_2\lambda^p\Phi(t,s) \le \Phi(\lambda t,s) \le K_2\lambda^q\Phi(t,s). \tag{7}$$

for $t \ge b(s)$, $\lambda \ge 1$.

Indeed (2) and (3) imply for $t \ge b(s)$ and for any $\lambda \ge 1$:

$$\begin{aligned}
&\Phi(\lambda t,s) \ge k\lambda^p\left(\Phi(t,s) - \frac{1}{2}\Phi(b_1(s),s)\right) \ge \frac{k}{2}\lambda^p\Phi(t,s), \\
&\Phi(\lambda t,s) \le K\lambda^q(\Phi(t,s) + \Phi(B_1(s),s)) \le 2K\lambda^q\Phi(t,s).
\end{aligned} \tag{8}$$

Consider $\overline{\Phi}(x,s) = \int_0^x \varphi(t,s)t^{p-1}dt$, where

$$\varphi(t,s) = \begin{cases} \dfrac{\Phi(b(s),s)}{b^{p+1}(s)}t, & 0 \le t \le b(s) \\ \sup\limits_{b(s)\le y\le t} \dfrac{\Phi(y,s)}{y^p(s)}, & t > b(s). \end{cases} \tag{9}$$

We will prove that $\overline{\Phi}$ satisfies (6) and is equivalent to Φ.

For the constants in (8) do not depend on $s \in \Omega$ to get (6) it is enough to prove that for any Orlicz function Ψ the inequality

$$k_2 \lambda^p \Psi(t) \le \Psi(\lambda t) \le K_2 \lambda^q \Psi(t), \quad \text{for all} \quad t \ge b, \lambda \ge 1 \tag{10}$$

implies the existence of positive constants k_1 and K_2, depending only on k_2, K_2, p and q such that for any $\lambda \ge 1$, $t \ge 0$ the inequality holds:

$$k_2 \lambda^p \overline{\Psi}(t) \le \overline{\Psi}(\lambda t) \le K_2 \lambda^q \overline{\Psi}(t), \tag{11}$$

where $\overline{\Psi}(t) = \int_0^t \Psi(u) u^{p-1} du$ with

$$\Psi(t) = \begin{cases} \dfrac{\Psi(b)}{b^{p+1}} t, & 0 \le t \le b \\ \sup\limits_{b \le y \le t} \dfrac{\Psi(y)}{y^p}, & t > b. \end{cases}$$

Obviously

$$\overline{\Psi}(t) = \frac{\Psi(b)}{b^{p+1}} \frac{t^{p+1}}{p+1}, \quad 0 \le t \le b; \tag{12}$$

$$\overline{\Psi}(t) = \frac{\Psi(b)}{p+1} + \int_b^t \sup_{b \le y \le x} \frac{\Psi(y)}{y^p} x^{p-1} dx, \quad t > b. \tag{13}$$

By using (10) we easily get for any $\lambda \ge 1$ the following inequalities that we will need in the sequel:

$$\sup_{b \le y \le \lambda t} \frac{\Psi(y)}{y^p} \le \frac{1}{k_2} \frac{\Psi(\lambda t)}{(\lambda t)^p}; \tag{14}$$

$$\sup_{\lambda b \le y \le \lambda t} \frac{\Psi(y)}{y^p} = \sup_{b \le y \le t} \frac{\Psi(\lambda y)}{(\lambda y)^p} \le K_2 \lambda^{q-p} \sup_{b \le y \le t} \frac{\Psi(y)}{y^p}. \tag{15}$$

Consider separately the cases:
I case: $0 \le t \le \lambda t \le b$ Obviously (12) implies $\overline{\Psi}(\lambda y) = \lambda^{p+1} \overline{\Psi}(t)$, i.e. (6) with $k_1 = K_1 = 1$.
II case: $0 \le t \le b < \lambda t$. Now (12) and (13) imply:

$$\begin{aligned} \overline{\Psi}(\lambda t) &= \tfrac{\Psi(b)}{p+1} + \int_b^{\lambda t} \Psi(u) u^{p-1} du \ge \frac{\Psi(b)}{p+1} + \frac{\Psi(b)}{b^p} \frac{\lambda^p t^p - b^p}{p+1} \\ &\ge \lambda^p \frac{\Psi(b)}{p+1} \left(\frac{t}{b} \right)^{p+1} = \lambda^p \overline{\Psi}(t). \end{aligned}$$

On the other hand using (14), (10) and (12) we get

$$\begin{aligned}
\overline{\Psi}(\lambda t) &\leq \frac{\Psi(b)}{p+1} + \frac{1}{k_2}\frac{\Psi(\lambda t)}{(\lambda t)^p}\int_b^{\lambda t} u^{p-1}du \leq \frac{\Psi(b)}{p+1} + \frac{1}{k_2}\frac{\Psi(\lambda t)}{p} \\
&\leq \left(\frac{1}{p+1} + \frac{1}{k_2 p}\right)\Psi(\lambda b) \leq K_2\left(\frac{1}{p+1} + \frac{1}{k_2 p}\right)\lambda^q \Psi(b) \\
&\leq K_3 \lambda^{p+q+1}\frac{\Psi(b)}{b^{p+1}}\frac{t^{p+1}}{p+1} \leq K_3\lambda^{p+q+1}\overline{\Psi}(t),
\end{aligned}$$

where $K_3 = K_2\left(1 + \dfrac{p+1}{pk_2}\right)$.

III case: $0 \leq b < t \leq \lambda t$. First we put $\overline{\Psi}(\lambda t) = \int_0^{\lambda b}\Psi(u)u^{p-1}du + \int_{\lambda b}^{\lambda t}\Psi(u)u^{p-1}du$ and estimate the two integrals separately.

$$\int_0^{\lambda b}\Psi(u)u^{p-1}du = \frac{\Psi(b)}{p+1} + \int_b^{\lambda b}\Psi(u)u^{p-1}du \geq \frac{\Psi(b)}{p+1} + \frac{\Psi(b)}{b^p}\frac{\lambda^p b^p - b^p}{p+1}dt = \lambda^p\int_0^b \Psi(u)u^{p-1}du,$$

and

$$\int_{\lambda b}^{\lambda t}\Psi(u)u^{p-1}du = \lambda^p\int_b^t \Psi(\lambda u)u^{p-1}du = \lambda^p\int_b^t \sup_{b\leq y\leq \lambda u}\frac{\Psi(y)}{y^p}u^{p-1}du \geq \lambda^p\int_b^t\Psi(u)u^{p-1}du.$$

The last two inequalities give us:

$$\overline{\Psi}(\lambda t) \geq \lambda^p\overline{\Psi}(t).$$

As above, by using (14) and (15) we obtain

$$\begin{aligned}
\overline{\Psi}(\lambda t) &= \overline{\Psi}(\lambda b) + \lambda^p\int_b^t\Psi(\lambda u)u^{p-1}du \\
&\leq K_3\lambda^{p+q+1}\overline{\Psi}(b) + \lambda^p\int_b^t\left(\sup_{b\leq y\leq\lambda b}\frac{\Psi(y)}{y^p} + \sup_{\lambda b\leq y\leq \lambda u}\frac{\Psi(y)}{y^p}\right)u^{p-1}du \\
&\leq K_3\lambda^{p+q+1}\overline{\Psi}(b) + \lambda^p\int_b^t\left(\frac{1}{k_2}\frac{\Psi(\lambda b)}{(\lambda b)^p} + \sup_{\lambda b\leq y\leq\lambda u}\frac{\Psi(y)}{y^p}\right)u^{p-1}du \\
&\leq K_3\lambda^{p+q+1}\overline{\Psi}(b) + \lambda^p\left(\tfrac{1}{k_2}+1\right)K_2\lambda^{q-p}\textstyle\int_b^t\Psi(u)u^{p-1}du \\
&\leq K_3\lambda^{p+q+1}\left(\overline{\Psi}(b) + \textstyle\int_b^t\Psi(u)u^{p-1}du\right) = K_3\lambda^{p+q+1}\overline{\Psi}(t).
\end{aligned}$$

The inequality (10) is proved with $k_1 = 1$, $K_1 = \max\left\{1, K_2\left(1+\frac{p+1}{pk_2}\right)\right\}$. Therefore $\overline{\Phi}$ satisfies (6) with k_1, K_1, not depending on $s \in \Omega$, $\lambda \geq 1$ and $t \geq 0$.

Finally we show that $\overline{\Phi} \sim \Phi$.

By using (8) we have for $t \geq b(s)$:

$$\begin{aligned} \overline{\Phi}(t,s) &= \overline{\Phi}(b(s),s) + \int_{b(s)}^t \sup_{b(s)\leq y\leq u} \tfrac{\Phi(y,s)}{y^p} u^{p-1} du = \tfrac{\overline{\Phi}(b(s),s)}{p+1} + \tfrac{1}{k_2}\int_{b(s)}^t \tfrac{\Phi(u,s)}{u} du \\ &\leq \tfrac{1}{k_2}\tfrac{\Phi(t,s)}{t}(t-b(s)) + \tfrac{\Phi(b(s),s)}{p+1} \leq \tfrac{1}{k_2}\left(\Phi(t,s) + \tfrac{k_2}{p+1}\Phi(b(s),s)\right). \end{aligned} \tag{16}$$

As $\overline{\Phi}$ is increasing the inequality holds true for $0 \leq t < b(s)$ also.

On the other hand the Δ^q condition (3) implies that

$$\Phi(2t,s) \leq L\Phi(t,s) + G(s), \quad \text{for all} \quad t \geq 0, \tag{17}$$

where L is positive constant, while G is non-negative integrable over Ω function. Equivalently

$$\Phi(t/2,s) \geq \frac{1}{L}(\Phi(t,s) - G(s)). \tag{18}$$

Now we easily get for $t \geq b$:

$$\begin{aligned} \overline{\Phi}(t,s) &\geq \tfrac{\Phi(b(s),s)}{p+1} + \int_{b(s)}^t \tfrac{\Phi(u,s)}{u} du = \tfrac{\Phi(b(s),s)}{p+1} + \int_0^t \tfrac{\Phi(u,s)}{u} du - \int_0^{b(s)} \tfrac{\Phi(u,s)}{u} du \\ &\geq \tfrac{\Phi(b(s),s)}{p+1} + \int_{t/2}^t \tfrac{\Phi(u,s)}{u} du - \Phi(b(s),s) \geq \Phi(t/2,s) - \tfrac{p}{p+1}\Phi(b(s),s) \\ &\geq \frac{1}{L}(\Phi(t,s) - G(s)) - \frac{p}{p+1}\Phi(b(s),s), \end{aligned}$$

i.e.

$$\Phi(t,s) \leq L\overline{\Phi}(t,s) + G(s) + \frac{Lp}{p+1}\Phi(b(s),s).$$

Obviously $\Phi(t,s) \leq \Phi(b(s),s)$ for $t \leq b(s)$. Therefore

$$\Phi(t,s) \leq L\overline{\Phi}(t,s) + G_1(s), \quad \text{for all} \quad t \leq b(s),$$

where $G_1(s) = G(s) + \dfrac{Lp}{p+1}\Phi(b(s),s)$ is non–negative and integrable over Ω. □

Corollary 4.1 *Let Φ be a Musielak–Orlicz function with $1 \leq \alpha(\Phi) \leq \beta(\Phi) < \infty$ and let for some $k \in [1, \alpha(\Phi)]$:*
i) there exist a non–negative integrable function h and a positive constant c_0 such that:

$$\Phi(uv,s) \leq c_0 u^k(\Phi(v,s) + h(s))$$

for all $u \in [0,1]$, $v \in \mathbb{R}^+$ and a.a. $s \in \Omega$;
ii) $\lim\limits_{u\to 0} \dfrac{\Phi(u,s)}{u^k} = 0$ for a.a. $s \in \Omega$.

Then there exists a Musielak–Oricz function $\overline{\Phi} \sim \Phi$ satisfying i), ii) and the uniform Δ^{*p} and Δ^{p+q+1} conditions (6).

Proof: Without loss of generality we may assume $c_0 \geq 1$. Choose p with $\alpha(\Phi) \in (p, p+1)^2$ and $q < \beta(\Phi)$. Then Φ satisfies the Δ^{*p}–condition and Δ^q–condition. Consider now the Musielak–Orlicz function $\overline{\Phi}$ constructed in Theorem 4. Taking into account (16) and

$$\overline{\Phi}(t,s) = \frac{\Phi(b(s),s)}{p+1}\left(\frac{t}{b(s)}\right)^{p+1}, \quad 0 \leq t \leq b(s) \tag{19}$$

we can write

$$\overline{\Phi}(t,s) \leq \frac{1}{k_2}\left(\Phi(t,s) + \frac{k_2}{p+1}\frac{t^k}{b^k(s)}\Phi(b(s),s)\right).$$

Now for any $u \in [0,1]$ we get:

$$\overline{\Phi}(uv,s) \leq \frac{1}{k_2}\Phi(uv,s) + u^k\frac{\Phi(b(s),s)}{p+1}, \quad 0 \leq v \leq b(s);$$

$$\overline{\Phi}(uv,s) \leq \frac{1}{k_2}\Phi(uv,s) + \frac{c_0}{p+1}u^k(\Phi(v,s)+h(s)), \quad b(s) < v.$$

By using once more the property i) of Φ we obtain:

$$\overline{\Phi}(uv,s) \leq Cu^k(\Phi(v,s)+h_1(s)),$$

where $C = c_0(\frac{1}{k_2} + \frac{1}{p+1})$ and $h_1 = h(s) + \Phi(b(s),s)$. To show that $\overline{\Phi}$ satisfies i) it is enough simply to use the equivalence $\overline{\Phi} \sim \Phi$ in the last inequality. On the other hand $\overline{\Phi}$ obviously satisfies the condition ii) according to (19). □

We mention that a "better" result than Theorem 4 holds true:

Theorem 5 *Let Φ satisfies the Δ^{*p} and Δ^q conditions for some $1 \leq p \leq q < \infty$. Then there exists $\overline{\Phi} \sim \Phi$ and positive constants $k_1^{'}$, $K_1^{'}$, such that*

$$k_1^{'}\lambda^p\overline{\Phi}(t,s) \leq \overline{\Phi}(\lambda t,s) \leq K_1^{'}\lambda^q\overline{\Phi}(t,s) \tag{20}$$

for any $\lambda \geq 1$, $t \geq 0$, $s \in \Omega$, i.e. $\overline{\Phi}$ satisfies the uniform Δ^{*p} and Δ^q conditions simultaniously.

The proof is similar to that of Theorem 4, by considering $\overline{\Phi}(x,s) = \int_0^x \varphi(t,s)t^{p-1}dt$, where

$$\varphi(t,s) = \begin{cases} \dfrac{\Phi(b(s),s)}{b^q(s)}, & 0 \leq t \leq b(s) \\ \inf\limits_{b(s)\leq y\leq t}\dfrac{\Phi(y,s)}{y^q(s)}, & t > b(s). \end{cases} \tag{21}$$

The function $\overline{\Phi}$ in this case does not inherit the properties i) and ii) in Corollary 4.1 and we will not use it in the sequel.

[2] If Φ satisfies the $\Delta^{*\alpha(\Phi)}$–condition we simply choose $p = \alpha(\Phi)$

5 Proof of Theorem 1

Theorem 1 *Let Φ be a Musielak–Orlicz function with $1 \le \alpha(\Phi) \le \beta(\Phi) < \infty$ and let for some $k \in [1, \alpha(\Phi)]$:*
i) there exist non–negative integrable function h and positive constant c_0 such that:

$$\Phi(uv, s) \le c_0 u^k (\Phi(v, s) + h(s))$$

for all $u \in [0, 1]$, $v \in \mathbb{R}^+$ and a.a. $s \in \Omega$;
ii) $\lim\limits_{u \to 0} \dfrac{\Phi(u, s)}{u^k} = 0$ for a.a. $s \in \Omega$.
Then for any measure space (Ω, Σ, μ) with a positive measure there is an equivalent UG^k–smooth norm in $L_\Phi(\Omega, \Sigma, \mu)$.

Proof: Choose p with $\alpha(\Phi) \in [p, p+1)$ and $q \in (\beta(\Phi), +\infty)$ and consider the function $\overline{\Phi} \sim \Phi$ constructed in Corollary 4.1, which satisfies simultaneously the uniform Δ^{*p} and Δ^{p+q+1}. Put $\beta = p + q + 1$ and $\Phi_1(u, s) = \displaystyle\int_0^u \frac{\overline{\Phi}(t, s)}{t} dt$, $u \ge 0$. Obviously

$$\frac{\overline{\Phi}(u/2, s)}{2} \le \Phi_1(u, s) \le \overline{\Phi}(u, s), \tag{22}$$

i.e. $\Phi \sim \Phi_1$ at 0 and ∞. Denote

$$\rho(u, s) = \begin{cases} \dfrac{\Phi_1(u, s)}{u^k}, & u > 0 \\ 0, & u = 0 \end{cases}$$

and $f_\Phi^k(u, s) = \max\{\rho(t, s) : t \in [0, u]\}$, $u \ge 0$. We will prove first that $f_\Phi^k(u, s) \in F(c)$ for some positive constant c. Indeed

$$\overline{\Phi}(u, s) \le c 2^\beta \overline{\Phi}(u/2, s) \le c 2^{\beta+1} \Phi_1(u, s). \tag{23}$$

Let $0 \le a < b$, $d_s = \max\{u \in [0, b] : \rho(u, s) = f_\Phi^k(b, s)\}$. Obviously $f_\Phi^k(b, s) = f_\Phi^k(a, s)$ if $d_s \le a$. If $a < d_s$ by the convexity of Φ_1 and (22), (23) we get for some $\theta \in (0, 1)$

$$\begin{aligned} f_\Phi^k(b, s) - f_\Phi^k(a, s) &\le \frac{\Phi_1(d_s, s) - \Phi_1(a, s)}{d_s^k} = (d_s - a) \frac{\overline{\Phi}(a + \theta(d_s - a), s)}{d_s^k (a + \theta(d_s - a))}, \\ &\le (d_s - a) \frac{\overline{\Phi}(d_s, s)}{d_s^{k+1}} \le c 2^{\beta+1} (b - a) \frac{f_\Phi^k(b, s)}{b} \end{aligned}$$

and obviously $u \to f_\Phi^k(u, s)$ is nondecreasing, continuous for $u \ge 0$ and $f_\Phi^k(0, s) = 0$ and thus the space $L_N(\Omega, \Sigma, \mu)$ is UG^k–smooth on the unit sphere, where $N = I^k(f_\Phi^k)$.

It remains to show that Φ is equivalent to N.

$$N(u,s) \le |u|^k f_\Phi^k(|u|,s) \le N(2^k u, s)$$

$$\begin{aligned}
\frac{\overline{\Phi}(u/2,s)}{2} &\le \Phi_1(u,s) \le |u|^k f_\Phi^k(|u|,s) \\
&\le \max\left\{\left(\frac{|u|}{t}\right)^k \Phi_1(t,s) : t\in[0,|u|]\right\} \\
&\le \max\left\{\left(\frac{|u|}{t}\right)^k \overline{\Phi}(t,s) : t\in[0,|u|]\right\} \\
&\le c_0(\overline{\Phi}(u,s)+h(s))
\end{aligned}$$

and thus

$$\frac{1}{2}\overline{\Phi}\left(\frac{u}{2^{k+1}},s\right) \le N(u,s) \le c_0(\overline{\Phi}(u,s)+h(s)).$$

□

6 Nakano spaces

The function $\Phi(u,s)=u^{p(s)}$, $p:\Omega\to[1,\infty)$ is a special case of Musielak–Orlicz function called Nakano function with a parameter. The space $L_{p(s)}(\Omega,\Sigma,\mu)=L_\Phi(\Omega,\Sigma,\mu)$ is called Nakano function space. We will apply Theorem 1 in order to prove the following:

Theorem 6 *Let $p:\Omega\to\mathbb{R}^+$ and $1\le \underline{p}=\operatorname{essinf}_{s\in\Omega}\{p(s)\}\le \overline{p}=\operatorname{esssup}_{s\in\Omega}\{p(s)\}<\infty$. If $1\le k\le \underline{p}$ then for any σ–finite measure μ on Ω the Nakano function space $L_{p(s)}(\Omega,\Sigma,\mu)$ admits equivalent UG^k–smooth norm.*

Proof: As for a σ–finite measure μ on Ω the space $L_{p(s)}(\Omega,\Sigma,\mu)$ is isometric to $L_{p(s)}(\Omega,\Sigma,\mu)$ for a suitable probability measure ν, we may without loss of generality consider only the case $\mu(\Omega)<\infty$, Ω free of atoms.

Put $N(t,s)=\int_0^t \frac{N_1(u,s)}{u}du$, where

$$N_1(u,s)=\begin{cases} u^{p(s)+1}, & u\in[0,1] \\ u^{p(s)}, & u\ge 1. \end{cases}$$

It is not hard to check that $L_{p(s)}(\Omega,\Sigma,\mu)\cong L_N(\Omega,\Sigma,\mu)$. On the oder hand $\underline{p}=\alpha(N)\le\beta(N)<\infty$. To finish the proof it is enough to observe that N satisfies the conditions:

i) $N(uv) \leq u^k N(v)$ for $u \in [0,1]$, $v \in \mathbb{R}$ and a.a. $s \in \Omega$;

ii) $\lim\limits_{u\to 0} \dfrac{N(u)}{u^k} = 0$ for a.a. $s \in \Omega$. $\square$

If Ω is a countable union of atoms of equal mass then we get the Nakano sequence space $\ell_{\{p_n\}}$. An equivalent definition for $\ell_{\{p_n\}}$ is Definition 2.7

Lemma 6.1 *Every Nakano sequence space $\ell_{\{p_n\}}$ is embedded isometrically in some Nakano function space $L_{p(s)}(0,1)$ for a sutable function $p(s) : \Omega \to [1,\infty)$ with $\underline{p} = \inf\limits_{s\in\Omega} p(s) = \inf\limits_{n\in\mathbb{N}} p_n$.*

Proof: Let $\{p_n\}_{n=1}^{\infty}$, $p_n \geq 1$, $0 = a_0 < a_1 < \ldots < a_n < a_{n+1} < \ldots < 1$, $\lim\limits_{n\to\infty} a_n = 1$. Let $h_n = |a_{n+1} - a_n|^{-\frac{1}{p_n}}$, $p(s) = p_n$ for $s \in [a_n, a_{n+1}]$ and

$$x_n(s) = \begin{cases} h_n, & t \in [a_n, a_{n+1}] \\ 0, & t \notin [a_n, a_{n+1}]. \end{cases}$$

Then $\ell_{\{p_n\}}$ is isometric to the subspace of $L_{p(s)}(0,1)$ generated by $\{x_n(s)\}_{n=1}^{\infty}$. Indeed, if $\{c_n\} \in \ell_{\{p_n\}}$, i.e. $\sum |c_n|^{p_n} < \infty$ holds:

$$\begin{aligned} \int_0^1 \left| \frac{\sum\limits_{n=1}^{\infty} c_n x_n(s)}{\|x\|_{\ell_{\{p_n\}}}} \right|^{p(s)} ds &= \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} \left| \frac{c_n h_n}{\|x\|_{\ell_{\{p_n\}}}} \right|^{p_n} ds \\ &= \sum_{n=1}^{\infty} \left| \frac{c_n h_n}{\|x\|_{\ell_{\{p_n\}}}} \right|^{p_n} |a_{n+1} - a_n| = \sum_{n=1}^{\infty} \left| \frac{c_n}{\|x\|_{\ell_{\{p_n\}}}} \right|^{p_n} = 1. \end{aligned}$$

$\square$

Now we are ready to prove

Theorem 2 *Let $\{p_n\}_{n=1}^{\infty}$ be such that $1 \leq p = \liminf\limits_{n\to\infty} p_n \leq \limsup\limits_{n\to\infty} p_n < \infty$ and the set $A = \{n \in \mathbb{N} : p_n < p\}$ is finite. Then there exist an equivalent UG^p–smooth norm in $\ell_{\{p_n\}}$.*

Proof: First let us note that $\ell_{\{q_n\}} \cong \ell_{\{p_n\}}$ for

$$q_n = \begin{cases} p_n, & p_n \geq p \\ p, & p_n < p. \end{cases}$$

By the previous lemma $\ell_{\{q_n\}} \hookrightarrow L_{q(s)}[0,1]$ and $1 \leq p = \operatorname{essinf}_{s\in[0,1]} q(s) \leq \operatorname{esssup}_{s\in[0,1]} q(s) < \infty$. The result now follows directly from Theorem 6 $\square$

Remark 6.1 *Theorem 2 holds true and if the set $A = \{n \in \mathbb{N} : p_n < p\}$ is not finite but $\sum_{n\in A} C^{\frac{1}{p-p_n}} < \infty$ for some constant $0 < C < 1$. It follows from [2]. Moreover $\ell_{\{q_n\}} \cong \ell_{\{p_n\}}$, where*

$$q_n = \begin{cases} p_n & if \quad n \notin A \\ p & if \quad n \in A. \end{cases}$$

Now from Theorem 2 we get that $\ell_{\{p_n\}}$ admits equivalent UG^p norm.

Remark 6.2 *If $p_n \geq p$, p odd integer and $\lim_{n\to\infty} p_n = p$ by [11] the space $\ell_{\{p_n\}}$ is UF^{p-1}–smooth and there is no p–times Fréchet differentiable bump in $\ell_{\{p_n\}}$. On the other hand Theorem 2 implies the existence of equivalent UG^p–smooth norm. The same remains true for p even if $\ell_{\{p_n\}}$ is not isomorphic to ℓ_p, which is F^∞–smooth.*

References

[1] N. Bonic, J. Frampton, Smooth functions on Banach manifolds. *J. Math. Mech.*, **15** (1966), 877-898.

[2] O. Blasco, P. Gregori, Type and cotype in Nakano sequence spaces $\ell_{(\{p_n\})}$.

[3] R. Deville, Geometrical implications of the existence of very smooth bump functions in Banach spaces. *Isr. J. Math. Mech.*, **67** (1989), 1-22.

[4] R. Deville, A charactarization of C^∞–smooth Banach spaces. *Proc. London. Math. Soc.*, **22** (1990), 13-17.

[5] R. Deville, G. Godfroy, V. Zizler, Smoothness and renorming in Banach spaces. Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Longman House, Brint Mill, Harlow, 1993. *Studia Math.*, **16** (1957), 173-182.

[6] A. Kaminska Indices, Convexity and Concavity in Musielak–Orlicz Spaces *Functiones et Approximatio*, **XXVI** (1998), 67-84.

[7] E. Katirtzoglou Type and Cotype of Musielak–Orlicz sequence Spaces *Journal of Mathematical Analysis and Applications*, **226** (2) (1998), 431-455.

[8] R. Maleev. Higher Order Uniformly Gateaux Differentiable Norms on Orlicz Spaces. *Rocky Mountain J. of Math*, **25** (3) (1995), 1117-1136.

[9] R. Maleev, S. Troyanski. Smooth funcions in Orlicz Spaces. *Cantemp. Math.* , **85** (1989), 355-370.

[10] R. Maleev, S. Troyanski. Smooth norms in Orlicz Spaces. *Canad. Math. Bull.*, **34** (1991), 74-82.

[11] R. Maleev, B. Zlatanov. Smoothness in Musielak–Orlicz Sequence Spaces. *Campte rendus de l'Académie des Sciences*, **55** (2002), 11-16.

[12] K. Sundaresan. Smooth Banach spaces. *Math. Ann.*, **173** (1967), 191-199.

[13] S. Troyanski. Gateaux differentiable norms in L_p. *Math. Ann.* , **287** (1990), 221-227.

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