Gâteaux Differentiability of Bump Functions in Separable Banach Spaces

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ABSTRACT It is shown that in large class of separable Banach spaces with an unconditional basis there is no 2–times Gâteaux differentiable bump functions. This result is applied in Orlicz and Lorentz sequence spaces. A condition for nonexistence of 2–times Gâteaux differentiable bump functions in Orlicz and Lorentz sequence spaces is found.

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1 Introduction

For many problems in the Geometry of Banach spaces and Nonlinear analysis in Banach spaces it is of importance the existence of bump functions with prescribed order of smoothness. One of the few results about higher order Gâteaux differentiability in separable Banach spaces is due to M. Fabian, Whitfield and V. Zizler. They have shown in [1] that there is no equivalent 2–times Gateaux differentiable norm in ℓ_p, $1 \leq p < 2$. F. Hernandes and S. Troyanski have shown that in Banach spaces satisfying a lower p–estimate for $1 \leq p < 2$ the order of Gâteaux smoothness can be only slightly better than the Fréchet one [2].

This paper is devoted to the existence of Gâteaux differentiable bump functions in Orlicz and Lorentz spaces. We begin with some definitions. Let $k \in \mathbb{N}$, $\omega : (0,1] \to \mathbb{R}^+$ be an increasing function with $\lim_{t\to 0} \omega(t)/t^k < \infty$. Let U be an open set in a Banach space X and $f : X \to \mathbb{R}$ be continuous. We shall say that $f \in G_{\omega,k}(U)$, if for any $x \in U$, $y \in X$ the representation

$$f(x+ty) = f(x) + \sum_{i=1}^{k} \frac{t^i}{i!} f^{(i)}(x)(y^i) + R_f^k(x,y,t),$$

holds, where $f^{(i)}(x)$, $i = 1, \ldots, k$ are i – linear bounded symmetric forms on X and

$$\lim_{t\to 0} \frac{|R_f^k(x,y,t)|}{\omega(|t|)} \leq c(x,y) < \infty.$$

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The main tool in the paper will be

Stegall Variational Principle: *Let X be a Banach space with the Radon–Nikodym property. Let $\varepsilon > 0$ and $\varphi : X \to \mathbb{R}\cup\{\infty\}$ be a lower semicontinuous function, bounded below. Assume that $D(\varphi) = \{x \in X : \varphi(x) < \infty\} \neq \emptyset$ and there exist $a > 0$ and $d \in \mathbb{R}$ such that for every $x \in X$,*

$$\varphi(x) \geq 2a\|x\| + d.$$

Then there exist $x_0 \in D(\varphi)$ and $f \in X^*$ with $\|f\| < \varepsilon$ such that for every $x \in X$,

$$\varphi(x) \geq \varphi(x_0) - f(x - x_0),$$

i.e. $\varphi + f$ attains its minimum at x_0.

This version of Stegall variational principle is due to Fabian (see e.g. [6], p. 88).

2 General result

Theorem 1 *Let X be a Banach space with the Radon-Nikodym property and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\liminf_{t\to 0} \omega(t)/t = 0$. Suppose that there exists an equivalent norm $|\cdot|$ on X such that for every $x \in X$ there exist $y \in X$ and sequences $t_m \to 0$ and $\tau_m \to 0$ fulfilling*

$$\lim_{m\to\infty} \omega(t_m)/\tau_m = 0 \text{ and } |x \pm t_m y| \geq |x| + \tau_m.$$

Then there exists no bump $b \in G_{\omega,1}(X)$.

Proof. Suppose b is a continuous bump, $b \in G_{\omega,1}(X)$. We assume $b(x) = 0$ for $\|x\| \geq 1$. Put $\delta(x) = b^{-2}(x)$. Then for every $x, y \in X$, $b(x) \neq 0$ we have

$$\limsup_{t\to 0} |R^1_\delta(x, y, t)|/\omega(|t|) = c(x, y) < \infty. \tag{1}$$

Let

$$\varphi(x) = \begin{cases} \delta(x) - |x| + 2, & b(x) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously $\varphi(x) \geq |x|$ for all $x \in X.X$ has the Radon–Nikodym property. Applying Stegall variational principle for φ we get $x_0 \in X$, $b(x_0) \neq 0$ and $f \in X^*$ such that for any $y \in X$ the inequality holds:

$$\varphi(x_0 + y) \geq \varphi(x_0) - f(y). \tag{2}$$

For suitable sign of t_n Taylor's formula gives:

$$(3)\qquad \begin{aligned} R^1_\delta(x_0,y,t_n) &= \delta(x_0+t_ny)-\delta(x_0)-t_n\delta^{'}(x_0)(y) \\ &\geq \varphi(x_0+t_ny)-\varphi(x_0)+|x_0+t_ny|-|x_0|-t_n\delta^{'}(x_0)(y) \\ &\geq |x_0+t_ny|-|x_0|-t_n((\delta^{'}(x_0)+f)(y)) \\ &\geq |x_0+t_ny|-|x_0|\geq \tau_n \end{aligned}$$

Obviously (3) implies

$$\lim_{n\to\infty}\frac{|R^1_\delta(x_0,y,t)|}{\omega(|t_n|)}\geq \lim_{n\to\infty}\frac{\tau_n}{\omega(|t_n|)}=\infty,$$

which contradicts $\delta\in G_{w,1}(X)$. Theorem 1 is proved. $\square$

In the next two sections we apply this result in Orlicz and Lorentz spaces.

3 Orlicz spaces

We recall that M is called an Orlicz function, provided M is even, convex function with $M(0)=0$, $M(t)>0$ for any $t\neq 0$. The Orlicz sequence space ℓ_M is the space of all sequences $x=\{x_n\}_{n=1}^\infty$ such that

$$\widetilde{M}(x/\lambda)=\sum_{n=1}^\infty M(x_n/\lambda)<\infty$$

for some $\lambda>0$, endowed with the norm:

$$\|x\|=\inf\{\lambda>0:\widetilde{M}(x/\lambda)\leq 1\}.$$

According to [5] the Boyd indices of ℓ_M are determined by:

$$\alpha_M=\sup\{p:\sup\{M(uv)/u^pM(v):u,v\in(0,1]\}<\infty\},$$

$$\beta_M=\inf\{p:\inf\{M(uv)/u^pM(v):u,v\in(0,1]\}>0\}.$$

We consider only spaces generated by Orlicz function M satisfying the Δ_2-condition at 0, i.e. $\beta_M<\infty$, which implies of course

$$(4)\qquad M(uv)\geq u^qM(v),\ \ u,v\in[0,1]$$

for some $q>\beta_M$ (see [5], p.140).

Using the method from [2] we prove the following:

Theorem 2 *Let ℓ_M be an Orlicz sequence space with M satisfying the Δ_2-condition at 0. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be such that*

$$\sup_{u,v\in(0,1]} \frac{M(uv)}{\omega(u)M(v)} = \infty.$$

Then there exists no bump $b \in G_{\omega,1}(\ell_M)$

Proof. Let $b \in G_{\omega,1}(\ell_M)$. Then the function $\beta(x_1, x_2) = b(x_1)b(x_2)$, defined on $\ell_M \times \ell_M$ is also in the class $G_{\omega,1}$. We show that this is not possible.

In view of Theorem 1 it suffices to find an equivalent norm $|\cdot|$ on $\ell_M \times \ell_M$ such that for any $x \in \ell_M \times \ell_M$ there exist $y \in \ell_M \times \ell_M$ and sequences $t_m \to 0$ and $\tau_m \to 0$ such that

$$\lim_{m\to\infty} \omega(t_m)/\tau_m = 0 \text{ and } |x \pm t_m y| \geq |x| + \tau_m.$$

Let $x = (x_1, x_2) \in \ell_M \times \ell_M$. We denote $\underbrace{g_k} = |x_1\underbrace{(k)}| + |x_2\underbrace{(k)}|$ and $\hat{x} = \{g_k(x)\}_{k=1}^{\infty}$. Then we have $\hat{x} \in \ell_M$. Indeed, $\widetilde{M}(\hat{x}) \leq \widetilde{M}(2x_1) + \widetilde{M}(2x_2) < \infty$. Set $|x|_{\ell_M\times\ell_M} = \|\hat{x}\|_{\ell_M}$. We shall show $|\cdot|$ is an equivalent norm in $\ell_M \times \ell_M$. Let us denote $\|\cdot\|_\infty = \max\{\|x_1\|, \|x_2\|\}$. First, obviously $|x| \geq \|x\|_\infty$ for any $x \in \ell_M \times \ell_M$. On the other hand

$$\begin{aligned}
\widetilde{M}\left(\frac{\hat{x}}{2\|x\|_\infty}\right) &= \sum_{k=1}^{\infty} M\left(\frac{|x_1(k)|}{2\|x\|_\infty} + \frac{|x_2(k)|}{2\|x\|_\infty}\right) \\
&\leq \frac{1}{2}\left(\sum_{k=1}^{\infty}\left(M\left(\frac{|x_1(k)|}{\|x\|_\infty}\right) + M\left(\frac{|x_2(k)|}{\|x\|_\infty}\right)\right)\right) \\
&\leq \frac{1}{2}\left(\sum_{k=1}^{\infty} M\left(\frac{|x_1(k)|}{\|x_1\|}\right) + \sum_{k=1}^{\infty} M\left(\frac{|x_2(k)|}{\|x\|_2}\right)\right) \leq 1,
\end{aligned}$$

by the convexity of M. Hence $|x| \leq 2\|x\|_\infty$ and $|\cdot|$ is an equivalent norm on $\ell_M \times \ell_M$.

Let now $x = (x_1, x_2) \in \ell_M \times \ell_M$. Without loss of generality we may suppose $|x| = 1$. We can choose sequences $t_n \to 0$, $v_n \to 0$ such that

$$\frac{M(t_n v_n)}{\omega(t_n)M(v_n)} > 4^n \text{ and } M(v_n) < 2^{-n}. \tag{5}$$

Set $m_n = [2^{-n}M(v_n)^{-1}]+1$, where $[a]$ is the largest integer not bigger than a. Choose a sequence $(k_n) \subseteq \mathbb{N}$ such that $k_n \geq k_{n-1} + m_{n-1}$ and $g_k(x) <$

$M(t_nv_n)/4$ for every $k \geq k_n$. Set

$$\begin{aligned} a_{1j} &= -a_{2j} = 1, \text{ if } x_1(j)x_2(j) \geq 0, \\ a_{1j} &= a_{2j} = 1, \text{ if } x_1(j)x_2(j) < 0, \end{aligned}$$

and $y_{in} = \sum_{j=k_n}^{k_n+m_n-1} v_n a_{ij} e_j$, $y_i = \sum_{n=1}^{\infty} y_{in}$, for $i = 1, 2$ and $y = (y_1, y_2)$. By (5) it follows easily that $y \in \ell_M \times \ell_M$.

By the construction of $a_{i,j}$ and y we have that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$g_k(x \pm ty) \geq g_k(x).$$

Hence

$$\begin{aligned} \widetilde{M}(\widehat{x \pm t_n y}) - \widetilde{M}(\hat{x}) &\geq \sum_{j=k_n}^{k_n+m_n-1} (M(g_j(x \pm t_n y)) - M(g_j(x))) \\ &\geq \sum_{j=k_n}^{k_n+m_n-1} \Big(M(\max_{i=1,2}\{|x_i(j) + a_{ij} t_n v_n|\}) - M(g_j(x)) \Big) \\ &\geq \sum_{j=k_n}^{k_n+m_n-1} (M(t_n v_n) - M(t_n v_n)/2) = \frac{m_n M(t_n v_n)}{2}. \end{aligned}$$

Thus

$$\widetilde{M}(\widehat{x \pm t_n y}) - \widetilde{M}(\hat{x}) \geq \frac{M(t_n v_n)}{2^{n+1} M(v_n)}. \tag{6}$$

On the other hand for sufficiently large n the inequality

$$0 < d_n = \|\widehat{x \pm t_n y}\| - 1 \leq 1$$

is fulfilled. Hence using (4) we establish

$$\begin{aligned} \widetilde{M}(\widehat{x \pm t_n y}) - \widetilde{M}(\hat{x}) &= \widetilde{M}\left(\|\widehat{x \pm t_n y}\| \frac{\widehat{x \pm t_n y}}{\|\widehat{x \pm t_n y}\|} \right) - 1 \\ &\leq \|\widehat{x \pm t_n y}\|^q - 1 = (1 + d)^q - 1 \leq q 2^{q-1} d_n, \end{aligned} \tag{7}$$

for some $q > \beta_M$.

Combining (6) and (7) we obtain

$$|x \pm t_n y| - 1 = \|\widehat{x \pm t_n y}\| - 1 \geq c M(t_n v_n)/2^{n+1} M(v_n), \tag{8}$$

where $c = (q 2^{q-1})^{-1}$.

Now setting $\tau_n = 2^n\omega(t_n)$ we have $\omega(t_n)/\tau_n \to 0$ as $n \to \infty$ and using (5), (6) and (8) we obtain

$$|x \pm t_n y| - |x| \geq c\frac{M(t_n v_n)}{2^{n+1}M(v_n)} \geq c\frac{4^n\omega(t_n)}{2^{n+1}} \geq c\tau_n,$$

which is exactly what we wanted. By Theorem 1 this contradicts the existence of bump $b \in G_{\omega,1}(\ell_M \times \ell_M)$. Theorem 2 is proved. $\square$

Remark Theorem 2 shows that if

$$\sup\{M(uv)/u^2M(v) : u, v \in (0,1]\} = \infty \tag{9}$$

there is no bump b with $\varphi(t) = b(x + ty)$ twice differentiable for every $x, y \in \ell_M$.

Using for $k = 2$

Theorem 3 *([3] Theorem 1) Let M be an Orlicz function that satisfies:*
1) $1 \leq k \leq \alpha_M \leq \beta_M < \infty$
2) $M(uv) \leq c_0 u^k M(v)$, $u \in [0,1]$, $v \in \mathbb{R}^+$ *for some positive* c_0
3) $\lim_{u\to 0} M(u)/u^k = 0$.
For any measure space $(S, \sum, \mu)$, μ a positive measure, in $X = L_M(S, \sum, \mu)$ there is an equivalent uniformly Gâteaux smooth norm.

We see that the estimates for the order of smoothness of bump functions in separable Orlicz sequence spaces with $1 < \alpha_M \leq 2$ found in Theorem 2 are exact.

4 Lorentz spaces

Let $1 \leq p < \infty$ and $w = \{w_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of positive scalars such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$.

We denote by $d(w, p)$ the Lorentz space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ for which

$$\|x\| = \sup\left\{\sum_{n=1}^{\infty} w_n |x_{\pi(n)}|^p\right\}^{1/p} < \infty \tag{10}$$

where the supremum is taken over all permutations π of the $\mathbb{N}$. From (10) we deduce that there exists a sequence rearrangement of the natural numbers $\{\pi(n)\}_{n=1}^{\infty}$ such that

$$\|x\| = \left\{\sum_{n=1}^{\infty} w_n |x_{\pi(n)}|^p\right\}^{1/p}.$$

The space $d(w,p)$ is a Banach space (see [5] p. 175).

Theorem 4 *Let be given a Lorentz space $d(w,p)$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(t) = \mu(t)t^p$, where $\liminf_{t\to 0}\mu(t) = 0$. Then there exists no bump $b \in G_{\omega,1}(d(w,p))$.*

Proof. Let $x = (x^1, x^2) \in d(w,p) \times d(w,p)$, $g_k(x) = |x^1_{\pi_1(k)}| + |x^2_{\pi_2(k)}|$, where $\{|x^i_{\pi_i(k)}|\}_{k=1}^\infty$ is a nonincreasing rearrangement of $\{|x^i_k|\}_{k=1}^\infty$. Denote $\hat{x} = \{g_k(x)\}_{k=1}^\infty$. Obviously $\hat{x} \in d(w,p) \times d(w,p)$. Define a norm in $d(w,p) \times d(w,p)$ by:

$$|x|_{d(w,p)\times d(w,p)} = \|\hat{x}\|_{d(w,p)}$$

Let assume that $|x|_{d\times d} = 1$

For a simplification of the notations we can assume that $\pi_i = id$, $i = 1, 2$.

We can choose sequences $\{t_n\}_{n=1}^\infty$, $\{k_n\}_{n=1}^\infty$ with the following properties

$$\mu(t_n) \le \frac{1}{4^n}, \quad 2^n\omega(t_n) \searrow 0,$$

$$\sum_{k_n+1}^{\infty} w_i|g_i(x)|^p < 2^n\omega(t_n),$$

$$\sum_{k_{n-1}+1}^{k_n} w_i \le \sum_{k_n+1}^{k_{n+1}} w_i.$$

Put $u_n = \left(2.2^n\mu(t_n)/\sum_{k_n+1}^{k_{n+1}} w_i\right)^{1/p}$. Obviously $u_n > u_{n+1}$. Now we choose a_{ij}:

$$\begin{aligned} a_{1j} &= -a_{2j} = 1, \text{ if } x_1(j)x_2(j) \ge 0, \\ a_{1j} &= a_{2j} = 1, \text{ if } x_1(j)x_2(j) < 0. \end{aligned}$$

Put $y_{in} = \sum_{j=k_n}^{k_{n+1}-1} u_n a_{ij} e_j$, $j = 1, 2$, $y_n = (y_{1n}, y_{2n})$, $y = \sum_{n=1}^\infty y_n$. Then

$$|y|^p = \|\hat{y}\|^p = 2\sum_{n=1}^\infty u_n^p \sum_{j=k_n}^{k_{n+1}-1} w_j \le 2\sum_{n=1}^\infty 1/2^n \le 2,$$

which ensures that $y \in d(w,p) \times d(w,p)$.

The definition of y_n gives us that $g_k(x \pm ty) \geq g_k(x)$ and we obtain

$$\begin{aligned}
|x \pm t_n y|^p &\geq |x \pm t_n y_n|^p = \|\widehat{x \pm t_n y_n}\|^p \\
&= \|\widehat{x}\|^p - \sum_{j=k_n}^{k_{n+1}-1} w_j |g_j(x)|^p + \sum_{j=k_n}^{k_{n+1}-1} w_j |g_j(x \pm t_n y_n)|^p \\
&\geq 1 - 2^n \omega(t_n) + \sum_{j=k_n}^{k_{n+1}-1} w_j |2t_n u_n - g_j(x)|^p \\
&\geq 1 - 2^n \omega(t_n) - \sum_{j=k_n}^{k_{n+1}-1} w_j |g_j(x)|^p + \sum_{j=k_n}^{k_{n+1}-1} w_j |2t_n u_n|^p \\
&\geq 1 - 2.2^n \omega(t_n) + 2.2^p.2^n \mu(t_n) t_n^p \geq 1 + c.2^n \omega(t_n)
\end{aligned}$$

Now put $\tau_n = 2^n \omega(t_n) \searrow 0$. Obviously

$$\lim_{n \to \infty} \frac{\omega(t_n)}{\tau_n} = 0$$

and

$$|x \pm t_n y|^p - 1 \geq c\tau_n.$$

Using the technique from [4] it can be shown that:

$$|x \pm t_n y| - |x| \geq c\tau_n.$$

Now we can apply Theorem 1. □

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