Gâteaux Differentiability of Bump Functions in Banach Spaces

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Abstract: Upper estimates for the order of Gâteaux smoothness of bump functions in Orlicz spaces $\ell_M(\mathbf{\Gamma})$ and Lorentz spaces $d(w,p,\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, are obtained.

Key Words: Orlicz space, Lorentz space, smooth bump functions.

1 Introduction

The existence of higher order Fréchet smooth norms and bump functions and its impact on the geometrical properties of a Banach space have been subject to many investigations beginning with the classical results for L_p–spaces in [1], [7]. An extensive study and bibliography may be found in [2]. As any negative result on the existence of Gâteaux smooth bump function immediately applies to the problem of existence of Fréchet smooth bump functions and norms the question arises of estimating the best possible order of Gâteaux smoothness of bump function in a given Banach space. A variational technique (the Ekeland variational principle) was applied in (Proposition II.5.5 [2]) to show that in $\ell_1(\mathbf{\Gamma})$, $\mathbf{\Gamma}$–uncountable, there is no continuous Gâteaux differentiable bump function. Following the same idea and using Stegall's variational principle, an extension of this result to Banach spaces with uncountable unconditional basis was given in [6]. As an application it was shown that in $\ell_p(\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, there is no continuous p–times Gâteaux differentiable bump function when p is odd and there is no continuous $([p]+1)$–times Gâteaux differentiable bump function in the case $p \notin \mathbb{N}$. The same result was obtained independently in [12]. This is essentially different from the case $\ell_p(\mathbb{N})$, p-odd, where equivalent p–times Gâteaux differentiable and even uniformly Gâteaux differentiable norms are constructed (see [14] and [11] respectively). Moreover, from the above results form [6], [13], [14], it follows that $\ell_p(\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, p odd, represent a negative answer to the question whether the existence of a k–times Gâteaux differentiable bump function in each separable subspace of a Banach space X implies the existence of such a bump function in the whole space X. We mention that the analogous problem for Fréchet C^k–smooth bump functions (Problem V.3 [2]) is still unsolved. It turns out that the variational technique is useful in the case of spaces with uncountable symmetric basis too (Theorem 1 below). As examples, Orlicz spaces $\ell_M(\mathbf{\Gamma})$ and Lorentz spaces$d(w,p,\mathbf{\Gamma})$, $\mathbf{\Gamma}$ uncountable, are considered. Estimates for the order of Gâteaux smoothness of bump functions are obtained. As a corollary sharp estimates for the order of Gâteaux differentiability of continuous bump functions in $\ell_M(\mathbf{\Gamma})$ are found. It is worthwhile to mention that results about differentiability of bump functions in $\ell_p(\mathbf{\Gamma})$ cannot be used directly for $\ell_M(\mathbf{\Gamma})$ and $d(w,p,\mathbf{\Gamma})$. Indeed, in [5] it is proved that $\ell_p(A)$ is isomorphic to a subspace of $d(w,p,\mathbf{\Gamma})$ iff A is countable. On the other hand $\ell_M(\mathbf{\Gamma})$ for $M \sim t^p(1+|\log t|)^q$ at 0, $p \geq 1$, $q \neq 0$, contains an isomorphic copy of $\ell_p(A)$ iff A is countable.

Let U be an open set in the Banach space X and let $f: U \to \mathbb{R}$ be continuous. Following [6] we shall say that f is $G^0_{\omega,k}$–smooth, $k \in \mathbb{N}$, in U ($f \in G^0_{\omega,k}(U)$) for some $\omega: (0,1] \to \mathbb{R}^+$,

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$\lim_{t\to 0} t^{-k}\omega(t) = 0$ if for any $x \in U$, $y \in X$ the representation holds:

$$f(x+ty) = f(x) + \sum_{i=1}^{k} \frac{t^i}{i!} f^{(i)}(x)(y^i) + R_f^k(x,y,t),$$

where $f^{(i)}(x)$, $i = 1,2,\ldots,k$ are i–linear bounded symmetric forms on X and

$$\lim_{t\to 0} |R_f^k(x,y,t)|/\omega(|t|) = 0.$$

If $U = X$ we use the notation $G^0_{\omega,[p]}$ instead of $G^0_{\omega,[p]}(X)$ and G_k, $k \in \mathbb{N}$, for the set of continuous k–times Gâteaux differentiable functions on X, for which $\lim_{t\to 0} |R_f^k(x,y,t)|/\omega(|t|) = 0$.

In what follows, for sake of simplicity we use "bump" instead of "bump function."

2 Main Result

Let X have symmetric basis $\{e_\gamma\}_{\gamma\in\mathbb{N}}$ with symmetric constant 1 and $0 \neq z \in X$, $z = \sum_{i=1}^{\infty} u_i e_{\gamma_i}$, $\gamma_i \neq \gamma_j$ for $i \neq j$. A sequence $\{z_k\}_{k\in\mathbb{N}}$, $z_k = \sum_{i=1}^{\infty} u_i e_{\alpha_{i,k}}$, $\alpha_{i,k} \neq \alpha_{j,l}$ for $(i,k) \neq (j,l)$, $\alpha_{i,k} \in \mathbf{\Gamma}$, is called a block basis generated by the vector z.

Denote

$$\lambda_n(z) = \left\| \sum_{j=1}^{n} z_j \right\|.$$

Theorem 1 *Let X be a Banach space, let $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$, $\sharp\mathbf{\Gamma} > \aleph_0$ be a symmetric boundedly complete basis in X with symmetric constant 1, and let $k \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \lambda_n(z) n^{-1/k} = 0, \tag{1}$$

for every $z \in X$.

Let $\omega : [0,1] \to \mathbb{R}^+$ be such that for every $x \in X$ there exist $y \in X$, $\mathrm{supp} y \cap \mathrm{supp} x = \emptyset$, and sequence $t_n \searrow 0$ that satisfy the inequality

$$\|x + t_n y\| - \|x\| \geq \omega(t_n), \quad n \in \mathbb{N}.$$

Then in X there is no continuous:

(i) $G^0_{\omega,k}$–smooth bump when $\omega(t) = o(t^k)$;

(ii) $G^0_{\omega,k+1}$–smooth bump when $\omega(t) = o(t^{k+1})$, k even;

(iii) k–times Gâteaux differentiable bump if $\omega(t) = t^k$;

(iv) $(k+1)$–times Gâteaux differentiable bump if $\omega(t) = t^{k+1}$, k even.

3 Proof of Theorem 1

We prove (i) and (ii). The proof of (iii) and (iv) is essentially the same. Suppose b is a continuous bump, $b \in G^0_{\omega,k}(X)$ ($b \in G^0_{\omega,k+1}(X)$). WLOG we assume $b(x) = 0$ for $\|x\| \geq 1$. Put

$$\delta(x) = \begin{cases} b^{-2}(x), & b(x) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for every $x, y \in X$, $b(x) \neq 0$ we have

(2) $$\lim_{t\to 0} |R_f^k(x,y,t)|/\omega(|t|) = 0 \quad \left(\lim_{t\to 0} |R_f^{k+1}(x,y,t)|/\omega(|t|) = 0\right).$$

Let

$$\varphi(x) = \begin{cases} \delta(x) - \|x\| + 2, & b(x) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously $\varphi(x) \geq \|x\|$ for all $x \in X$. X has the Radon–Nicodym property. Applying Stegall variational principle for φ in X we get $x_0 \in X$, $b(x_0) \neq 0$ and $f \in X^*$ such that for any $y \in X$ the inequality holds:

(3) $$\varphi(x_0 + y) \geq \varphi(x_0) - f(y).$$

Let $y_0 \in X$, $\operatorname{supp} y_0 \cap \operatorname{supp} x_0 = \emptyset$ and $t_n \searrow 0$, be such that

(4) $$\|x_0 + t_n y_0\| + \|x_0 - t_n y_0\| - 2\|x_0\| \geq 2\omega(t_n)$$

Let $\{y_\alpha\}_{\alpha\in A}$, $\operatorname{supp} y_\alpha \cap \operatorname{supp} x_0 = \emptyset$, $\sharp A > \aleph_0$, be a block basis generated by y_0. As for every $\{\alpha_i\}_{i=1}^\infty \subset A$ the sequence $\{y_{\alpha_i}\}_{i=1}^\infty$ is symmetric basis in $\overline{\operatorname{span}\{y_{\alpha_i}\}_{i=1}^\infty}^{\|\cdot\|}$ satisfying (1), then

$$\lim_{i\to\infty} P(y_{\alpha_i}) = P(0)$$

provided P is polynomial, $\deg P \leq k$. This fact goes back to [1].

Then there exists $\alpha_0 \in A$ such that

$$\delta^{(j)}(x_0)(y_{\alpha_0}^j) = 0, \quad j = 1, 2, \ldots, k$$

and Taylor's formula gives

$$\delta(x_0 + t y_{\alpha_0}) = \delta(x_0) + R_\delta^k(x_0, y_{\alpha_0}, t)$$

$$\left(\delta(x_0 + t y_{\alpha_0}) = \delta(x_0) + \frac{t^{k+1}}{(k+1)!}\delta^{(k+1)}(x_0)(y_{\alpha_0}^{(k+1)}) + R_\delta^{(k+1)}(x_0, y_{\alpha_0}, t)\right).$$

Using 3 we obtain

(5) $$\begin{aligned} R_\delta^k(x_0, y_{\alpha_0}, \pm t) &= \delta(x_0 \pm t y_{\alpha_0}) - \delta(x_0) \geq \|x_0 \pm t y_{\alpha_0}\| - \|x_0\| \mp f(y_{\alpha_0}) \\ R_\delta^{(k+1)}(x_0, y_{\alpha_0}, \pm t) &\geq \|x_0 \pm t y_{\alpha_0}\| - \|x_0\| \mp t\left(f(y_{\alpha_0}) + \tfrac{t^{k+1}}{(k+1)!}\delta^{(k+1)}(x_0)(y_{\alpha_0}^{(k+1)})\right). \end{aligned}$$

Obviously (5) implies

(6) $$\begin{aligned} &|R_\delta^k(x_0, y_{\alpha_0}, t)| + |R_\delta^k(x_0, y_{\alpha_0}, -t)| \geq \|x_0 + t y_{\alpha_0}\| + \|x_0 - t y_{\alpha_0}\| - 2\|x_0\| \\ &\left(|R_\delta^{k+1}(x_0, y_{\alpha_0}, t)| + |R_\delta^{k+1}(x_0, y_{\alpha_0}, -t)| \geq \|x_0 + t y_{\alpha_0}\| + \|x_0 - t y_{\alpha_0}\| - 2\|x_0\|\right). \end{aligned}$$

As (4) implies

$$\|x_0 + t_n y_{\alpha_0}\| + \|x_0 - t_n y_{\alpha_0}\| - 2\|x_0\| \geq 2\omega(t_n)$$

it is easy to obtain from (6)

$$\frac{|R_\delta^k(x_0, y_{\alpha_0}, t_n)| + |R_\delta^k(x_0, y_{\alpha_0}, -t_n)|}{\omega(|t_n|)} \geq 2 > 0$$

$$\frac{|R_\delta^{k+1}(x_0, y_{\alpha_0}, t_n)| + |R_\delta^{k+1}(x_0, y_{\alpha_0}, -t_n)|}{\omega(|t_n|)} \geq 2 > 0,$$

which contradicts $\delta \in G^0_{\omega,k}(X)$ ($\delta \in G^0_{\omega,k+1}(X)$). Theorem1 is proved. □

In what follows we shall apply Theorem1 for Orlicz and Lorentz spaces, $\ell_M(\mathbf{\Gamma})$ and $d(w,p,\mathbf{\Gamma})$, $\mathbf{\Gamma}$–uncountable.

4 Orlicz Spaces $\ell_M(\mathbf{\Gamma})$

We recall that M is called an Orlicz function, provided M is an even, convex function with $M(0) = 0$, $M(t) > 0$ for any $t \neq 0$. The Orlicz space $\ell_M(\mathbf{\Gamma})$ is the space of all vectors $x = \{x_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ such that

$$\widetilde{M}(x/\lambda) = \sum_{\gamma\in\mathbf{\Gamma}} M(x_\gamma/\lambda) < \infty$$

for some $\lambda > 0$, endowed with the norm

$$\|x\| = \inf\{\lambda > 0 : \widetilde{M}(x/\lambda) \leq 1\}.$$

According to [10] the Boyd indices of ℓ_M are determined by

$$\alpha_M = \sup\{p : \sup\{M(uv)/u^pM(v) : u,v \in (0,1]\} < \infty\},$$

$$\beta_M = \inf\{p : \inf\{M(uv)/u^pM(v) : u,v \in (0,1]\} > 0\}.$$

We consider only spaces generated by Orlicz function M satisfying the Δ_2-condition at 0, i.e. $\beta_M < \infty$, which implies of course

(7) $$M(uv) \geq u^qM(v), \quad u,v \in [0,1]$$

for some $q > \beta_M$ (see [9], p.140).

Finally we mention that the unit vectors $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ form a symmetric basis of $\ell_M(\mathbf{\Gamma})$ with symmetric constant 1, which is boundedly complete.

For $g : (0,1] \to \mathbb{R}^+$ denote

$$d_M(g) = \sup\{M(uv)/g(u)M(v) : u,v \in (0,1]\}.$$

To apply Theorem 1 for $\ell_M(\mathbf{\Gamma})$ we need the following Lemmas.

Lemma 4.1 *Let $p \geq 1$ and M be Orlicz function satisfying the conditions*

(8) $$\lim_{t\to 0} M(t)/t^p = 0$$

$$d_M(t^p) = c < \infty. \tag{9}$$

Then every block basis $\{z_j\}_{j=1}^{\infty}$ *of the unit vector basis* $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ *in* $\ell_M(\mathbf{\Gamma})$*, generated by one vector, satisfies*

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} z_j \right\| n^{-1/p} = 0.$$

Proof: Let $z = \sum_{i=1}^{\infty} u_i e_{\gamma_i} \in \ell_M(\mathbf{\Gamma})$ and $\{e_{j,i}\}_{i=1}^{\infty}$, $j = 1, 2, \ldots$ be disjoint subsets of $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$. Then $z_j = \sum_{i=1}^{\infty} u_i e_{j,i}$, $j = 1, 2, \ldots$ form a symmetric block basis of $\{e_\gamma\}$, generated by z.

Denote $\mu(t) = M(t)/t^p$. From (8) and (9) it follows

$$\lim_{t\to 0} \mu(t) = 0, \tag{10}$$

$$\mu(t_1) \le c\lambda(t_2), \quad \text{provided} \quad t_1, t_2 \in (0, 1], \quad t_1 < t_2. \tag{11}$$

Let $\varepsilon > 0$ arbitrary. Find m such that

$$\sum_{i=m+1}^{\infty} M(u_i) < \varepsilon/2c.$$

From

$$1 = \widetilde{M}\left(\sum_{j=1}^{n} z_j \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right) = \sum_{j=1}^{n} \widetilde{M}\left(z_j \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right) = n\widetilde{M}\left(z \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right)$$

we obtain

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} z_j \right\|^{-1} = 0, \tag{12}$$

$$\left(\left\| \sum_{j=1}^{n} z_j \right\| n^{-1/p} \right)^p = \sum_{i=1}^{\infty} u_i^p \mu\left(u_i \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right). \tag{13}$$

Using (10) and (12) we obtain for large n

$$\sum_{i=1}^{m} u_i^p \mu\left(u_i \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right) < \varepsilon/2.$$

On the other hand (11) implies

$$\sum_{i=m+1}^{\infty} u_i^p \mu\left(u_i \Big/ \left\| \sum_{j=1}^{n} z_j \right\| \right) \le c \sum_{m+1}^{\infty} M(u_i) < \varepsilon/2$$

and we get from (13)

$$\left(\left\| \sum_{j=1}^{n} z_j \right\| n^{-1/p} \right)^p < \varepsilon$$

which completes the proof. □

Lemma 4.2 *Let $d_M(\omega) = \infty$. Then for every $x \in \ell_M(\mathbf{\Gamma})$ there exist $y \in \ell_M(\mathbf{\Gamma})$,* supp $y \cap$ supp $x = \emptyset$ *and a sequence $\{t_n\} \searrow 0$, such that*

$$\|x + t_n y\| \geq \|x\| + c\omega(t_n) \tag{14}$$

for some $c > 0$ and any $n \in \mathbb{N}$.

Proof: We note first that

$$\liminf_{t \to 0} \omega(t)/t = 0.$$

If $x = 0$ choose sequence $t_n \searrow 0$ such that $\lim_{n\to\infty} \omega(t_n)/t_n = 0$. Then (14) holds true trivially for any $y \neq 0$ with $c = \|y\| > 0$.

WLOG suppose $M(1) = 1$.

Fix arbitrary $x = \sum_{i=1}^{\infty} x_i e_{\gamma_i} \in \ell_M(\mathbf{\Gamma})$, $\|x\| = 1$ and choose sequences $\{t_n\}$, $\{v_n\}$ such that

$$t_n \searrow 0, \quad v_n \searrow 0, \quad t_n, v_n > 0; \tag{15}$$

$$M(t_n v_n)/\omega(t_n) M(v_n) > 2^n; \tag{16}$$

$$M(v_n) < 2^{-n}. \tag{17}$$

Put $m_n = [1/2^n M(v_n)] + 1$ and find $k_1 \in \mathbb{N}$ such that

$$M(x_i) < M(u_1 v_1)/2, \quad i \geq k_1.$$

Define inductively a sequence of naturals $\{k_n\}_{n=1}^{\infty}$ such that

$$k_{n-1} + m_{n-1} \leq k_n,$$

$$M(x_i) < M(t_n v_n)/2, \; i \geq k_n.$$

For a sequence $\{A_n\}_{n=1}^{\infty}$ of finite disjoint subsets of $\mathbf{\Gamma}$, such that $A_n \cap \operatorname{supp} x = \emptyset$, $\sharp A_n = k_n$, put

$$y_n = v_n \sum_{\gamma \in A_n} e_\gamma, \quad y = \sum_{n=1}^{\infty} y_n.$$

Obviously $\widetilde{M}(y) = \sum_{n=1}^{\infty} m_n M(v_n) \leq 2$, which secures $y \in \ell_M(\mathbf{\Gamma})$. We have supp $(x + ty) =$ $\mathrm{supp\ x} \cup \ (\bigcup_{\mathrm{n}=1}^{\infty} \mathrm{A_n})$ for any $t \neq 0$ and therefore

$$\widetilde{M}(x + t_n y) - \widetilde{M}(x) \geq \sum_{\gamma \in A_n} M(t_n v_n) = m_n M(t_n v_n) \geq M(t_n v_n)/2^n M(v_n). \tag{18}$$

Remove as many elements of the sequence $\{t_n\}$ as necessary to have

$$0 < d_n = \|x + t_n y\| - 1 \leq 1$$

and keep the same notation for the remaining sequence. Now (7) implies

$$\begin{aligned} \widetilde{M}(x + t_n y) - \widetilde{M}(x) &= \widetilde{M}\left(\|x + t_n y\| \frac{x + t_n y}{\|x + t_n y\|}\right) - 1 \\ &\leq \|x + t_n y\|^q - 1 = (1 + d_n)^q - 1 \leq q 2^{q-1} d_n, \end{aligned} \tag{19}$$

for some $q > \beta_M$.

Combining (18) and (19) we obtain

$$\|x + t_n y\| - 1 \geq cM(t_n v_n)/2^n M(v_n),$$

where $c = (q2^{q-1})^{-1}$.

Let $x \neq 0$ arbitrary. Find $\overline{y}$, supp $\overline{y} \cap$ supp $x = \emptyset$ such that

$$\left\| \frac{x}{\|x\|} + t_n \overline{y} \right\| - 1 \geq cM(t_n v_n)/2^n M(v_n).$$

Obviously for $y = \|x\| \overline{y}$ we have from (16)

$$\|x + t_n y\| - \|x\| \geq c\|x\| M(t_n v_n)/2^n M(v_n) \geq c\|x\|\omega(t_n).$$

which ends the proof. □

Theorem 2 *Let M be an Orlicz function, $M \not\sim t^n$ at 0 for $n \in \mathbb{N}$ and $\omega : (0,1] \to \mathbb{R}^+$, $d_M(\omega) = \infty$. Then*

(a) if $\alpha_M \notin \mathbb{N}$ in $\ell_M(\mathbf{\Gamma})$ there is no continuous $G^0_{\omega,[\alpha_M]}$–smooth bump;

(b) if $\alpha_M \in \mathbb{N}$ in $\ell_M(\mathbf{\Gamma})$ there is no continuous G^0_{ω,α_M}–smooth bump, provided $d_M(t^{\alpha_M}) < \infty$ and there is no continuous $G^0_{\omega,\alpha_M - 1}$–smooth bump, provided $d_M(t^{\alpha_M}) = \infty$.

The proof in all cases is straightforward, applying Lemma 4.1 for appropriate p and Lemma4.2.

Corollary 1 *Let M be Orlicz function, $M \not\sim t^n$ at 0 for $n \in \mathbb{N}$. If f is a k–times Gâteaux differentiable continuous bump in $\ell_M(\mathbf{\Gamma})$, then*

$$k \leq E_M = \begin{cases} [\alpha_M], & d_M(t^{[\alpha_M]}) < \infty \\ \alpha_M - 1, & \alpha_M \in \mathbb{N}, \ d_M(t^{\alpha_M}) = \infty. \end{cases}$$

Remark 1: The above estimates for the order of Gâteaux differentiability are sharp. Indeed, the statement is obviously true for $\alpha_M \notin \mathbb{N}$ or $\alpha_M \in \mathbb{N}$, $d_M(t^{\alpha_M}) = \infty$ because for such M in $\ell_M(\mathbf{\Gamma})$ there exist equivalent even E_M–times Fréchet differentiable norms (see [12]). On the other hand if $\alpha_M = k \in \mathbb{N}$, $d_M(t^k) < \infty$, then $\lim_{t\to 0} M(t)/t^k = 0$ (otherwise $M \sim t^k$ at 0). Following step by step the construction from [12] with the corresponding changes in Lemmas 2 and 4 one can construct Orlicz function $N \sim M$ at 0 such that $\widetilde{N}$ is k–times Gâteaux differentiable and for any $x, y \in \ell_N(\mathbf{\Gamma})$

$$\left| \widetilde{N}(x + ty) - \sum_{j=0}^{k} \frac{t^j}{j!} \widetilde{N}^{(j)}(y^j) \right| \leq \left(\widetilde{N}(x) + \widetilde{N}(y) \right) \Phi(t) + c\widetilde{N}(ty)$$

where $\Phi(t) = o(t^k)$ depends only on M. This implies immediately that the norm in $\ell_N(\mathbf{\Gamma})$ is k–times Gateaux differentiable in $\ell_N(\mathbf{\Gamma}) \backslash \{0\}$.

Remark 2: Using Lemma 4.1 for p even integer and the obvious inequality in $\ell_{p+1}(\mathbf{\Gamma})$,

$$\|x+ty\|+\|x-ty\|-2\|x\| \geq c(x,y)|t|^{p+1}, \quad \text{supp x} \cap \text{supp y} = \emptyset,$$

we easily get as a corollary of Theorem 1 that in $\ell_{p+1}(\mathbf{\Gamma})$, p even, there is no continuous $(p+1)$–times Gâteaux differentiable bump. This result was obtained independently in [13] and [6].

Remark 3: Some new results have been obtained recently for Orlicz spaces h_M, with $\alpha_M = \infty$. In [4], by using the fact that every separable isomorphically polyhedral Banach space has equivalent analytic norm [3], it was proved that $h_M(\mathbf{\Gamma})$, $\alpha_M = \infty$ has analytic norm iff $\mathbf{\Gamma}$ is countable and $h_M(\mathbf{\Gamma})$ is isomorphically polyhedral. On the other hand Leung [8] gave an example of an Orlicz function M for which $\alpha_M = \infty$ but h_M fails to be isomorphically polyhedral. The corresponding Orlicz space h_M is an example of a separable Banach space with Fréchet C^∞–smooth norm [12], which has no equivalent analytic norm.

5 Lorentz Spaces $d(w,p,\mathbf{\Gamma})$

Let $1 \leq p < \infty$ and let $w = \{w_n\}_{n=1}^\infty$ be a nonincreasing sequence of positive scalars such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^\infty w_n = \infty$.

We denote by $d(w,p,\mathbf{\Gamma})$ the Lorentz space of all real functions $x = x(\alpha)$ defined on the set $\mathbf{\Gamma}$ for which

$$\|x\| = \sup \left\{ \sum_{n=1}^\infty w_n |x(\alpha_n)|^p \right\}^{1/p} < \infty$$

where the supremum is taken over all sequences $\{\alpha_n\}_{n=1}^\infty$ of different elements of $\mathbf{\Gamma}$. There exist a sequence $\{\alpha_n^*\}_{n=1}^\infty$ such that $|x(\alpha_1^*)| \geq |x(\alpha_2^*)| \geq \ldots$, $\lim_{n\to\infty} x(\alpha_n^*) = 0$, $x(\alpha) = 0$ if $\alpha \neq \alpha_n^*$, $n = 1, 2, \ldots$ and

$$\|x\| = \left\{ \sum_{n=1}^\infty w_n |x(\alpha_n^*)|^p \right\}^{1/p}.$$

The space $d(w,p,\mathbf{\Gamma})$ is a Banach space and the canonical basis $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$ is a symmetric basis [9].

Lemma 5.1 *Let $p \geq 1$. For every $z \in d(w,p,\mathbf{\Gamma})$, $\lim_{n\to\infty} \lambda_n(z) n^{-1/p} = 0$.*

Proof: Let $z = \sum_{i=1}^\infty u_i e_{\gamma_i}$, where $|u_i| \geq |u_{i+1}|$, $i \in \mathbb{N}$ and $z_j = \sum_{i=1}^\infty u_i e_{j,i}$, where $\{e_{j,i}\}_{i=1}^\infty$, $j = 1, 2, \ldots$ are disjoint subsets of $\{e_\gamma\}_{\gamma\in\mathbf{\Gamma}}$. Obviously

$$\left\| \sum_{j=1}^n z_j \right\|^p = \left\| \sum_{j=1}^n \sum_{i=1}^\infty u_i e_{j,i} \right\|^p = \sum_{i=1}^\infty \sum_{j=n(i-1)+1}^{ni} w_j |u_i|^p.$$

Let $\varepsilon > 0$ arbitrary. Find m such that

$$\sum_{j=m+1}^\infty w_i |u_i|^p < \varepsilon/2. \tag{20}$$

It is not hard to observe that (20) implies

$$\sum_{i=m+1}^{\infty}\sum_{j=n(i-1)+1}^{ni} w_j|u_i|^p/n \le \sum_{i=m+1}^{\infty} w_{n(i-1)+1}|u_i|^p < \varepsilon/2. \tag{21}$$

On the other hand

$$\sum_{i=1}^{m}\sum_{j=n(i-1)+1}^{ni} w_j|u_i|^p/n \le \frac{w_1+\cdots+w_n}{n}\sum_{i=1}^{m}|u_i|^p.$$

Obviously $\lim_{n\to\infty}(w_1+\cdots+w_n)/n = 0$ and for n large enough

$$\sum_{i=1}^{m}\sum_{j=n(i-1)+1}^{ni} w_j|u_i|^p/n \le \varepsilon/2, \tag{22}$$

which together with (20) implies

$$\left(\left\|\sum_{j=1}^{n} z_j\right\| n^{-1/p}\right)^p < \varepsilon.$$

Lemma5.1 is proved. □

Lemma 5.2 *For every $x \in d(w,p,\mathbf{\Gamma})$ and every $\omega : (0,1] \to \mathbb{R}^+$, $\omega(t) = \mu(t)t^p$, $\lim_{t\to 0}\mu(t) = 0$, $\mu(t) \ge t$, there exist $y \in d(w,p,\mathbf{\Gamma})$,* supp y ∩ supp x = ∅ *and sequence $t_n \searrow 0$, such that*

$$\|x + t_n y\| \ge \|x\| + c(x)\omega(t_n). \tag{23}$$

Proof: Suppose $\|x\| = 1$ and let $\{x_i\}_{i=1}^{\infty}$ be the decreasing rearrangement of the nonzero coordinates of x. Choose decreasing sequence $\{t_n\}_{n=1}^{\infty}$, such that $\mu(t_n) \le 2^{-n}$ and increasing sequence of naturals $\{k_n\}_{n=1}^{\infty}$,

$$\sum_{i=k_n+1}^{\infty} w_i|x_i|^p < \omega(t_n),$$

$$\sum_{i=k_{n-1}+1}^{k_n} w_i \ge \sum_{i=k_{n-2}+1}^{k_{n-1}} w_i.$$

Put $u_n = \left(2\mu(t_n)/\sum_{i=k_n+1}^{k_{n+1}} w_i\right)^{1/p}$, $n = 1,2,\ldots$. Obviously $u_n > u_{n+1}$ for every $n \in \mathbb{N}$. Choose a sequence $\{A_n\}_{n=1}^{\infty}$ of disjoint subsets of $\mathbf{\Gamma}$, $A_n \cap \text{supp x} = \emptyset$, $\sharp A_n = k_{n+1} - k_n$ and consider the sequence $\{y_n\}_{n=1}^{\infty}$, $y_n = u_n\sum_{\gamma\in A_n} e_\gamma$. From

$$\sum_{n=1}^{\infty} u_n^p \sum_{j=k_n+1}^{k_{n+1}} w_j = 2\sum_{n=1}^{\infty}\mu(t_n) \le 2$$

it follows that $y = \sum_{i=1}^{\infty} y_n \in d(w,p,\mathbf{\Gamma})$.

Let us check that y satisfies (23). We have

$$\begin{aligned}\|x+t_ny_n\|^p &\geq \sum_{j=1}^{k_n} w_j|x_j|^p + t_n^p u_n^p \sum_{j=k_n+1}^{k_{n+1}} w_j = \sum_{j=1}^{k_n} w_j|x_j|^p + 2\omega(t_n) \\ &\geq \sum_{j=1}^{\infty} w_j|x_j|^p + \omega(t_n) \geq 1+\omega(t_n).\end{aligned}$$

Now from

$$\|x+t_ny\|^p = \left\| x + t_n \sum_{j=1}^{\infty} y_j \right\|^p \geq \|x+t_ny_n\|^p$$

we obtain

$$\|x+t_ny\|^p \geq 1+\omega(t_n).$$

Denote $\|x+t_ny\| = 1+d_n$. Remove those t_n for which $1 < |t_n|\|y\|$. Then from the inequality

$$(1+d_n)^p \leq 1 + p2^{p-1}d_n$$

it follows (23) with $c = (p2^{p-1})^{-1}$.

If $\|x\| \neq 1$ applying the above for $x/\|x\|$ we find $\overline{y}$ and $t_n \searrow 0$ such that

$$\|x + t_n\|x\|\ \overline{y}\| \geq \|x\| + c\|x\|\omega(t_n),$$

i.e. (23) with $y = \|x\|\overline{y}$ and $c(x) = c\|x\|$.

Lemma 5.2 is proved. □

Using Lemmas 5.1 and 5.2 in Theorem 1 we get:

Theorem 3 *Let $p \geq 1$, $w_n \searrow 0$, $\sum_{n=1}^{\infty} w_n = \infty$ and $\omega : [0,1] \to \mathbb{R}^+$ be such that $\omega(t) = o(t^p)$. Then in $d(w,p,\mathbf{\Gamma})$ there is no continuous $G^0_{w,[p]}$–smooth bump.*

Corollary 2 *Let $p \geq 1$, $w_n \searrow 0$, $\sum_{n=1}^{\infty} w_n = \infty$. If f is a continuous k–times Gâteaux differentiable bump in $d(w,p,\mathbf{\Gamma})$, then $k \leq [p]$.*

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