On Musielak–Orlicz Sequence Spaces with an Asymptotic ℓ_∞ dual[1]

B. Zlatanov

ABSTRACT: We investigate MO sequence spaces ℓ_Φ with a dual ℓ_Φ^*, which is stabilized asymptotic ℓ_∞ with respect to the unit vector basis. We give a complete characterization of the bounded relatively weakly compact subsets $K \subset \ell_\Phi$. We prove that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and thus ℓ_Φ fails the fixed point property for closed, bounded convex sets and non–expansive (or contractive) maps on them.

2000 Mathematics Subject Classification: 46B20, 46B45, 46E30, 46A45, 47H10.

Keywords: Mushielak–Orlicz sequence spaces, asymptotically isometric copy of ℓ_1, asymptotic ℓ_∞ space, fixed point property, weakly compact.

1 Introduction

The notion of asymptotic ℓ_p spaces first appeared in [14], where the collection of spaces that are now known as stabilized asymptotic ℓ_p spaces were introduced. Later in [13], more general collection of spaces, known as asymptotic ℓ_p spaces were introduced. Characterization of the stabilized asymptotic ℓ_∞ MO sequence space was given in [5]. It is found in [17] that if the dual of a MO sequence space ℓ_Φ is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis then ℓ_Φ is saturated with complemented copies of ℓ_1 and has the Schur property.

A characterization of the relatively weakly compact sets in an Orlicz spaces $L_M[0,1]$, such that the function N complementary to M satisfies $\lim_{t\to\infty} \frac{N(\lambda t)}{N(t)} = \infty$ for some $1 < \lambda < \infty$ is given in [2]. Using the technique of [2] and [17] we generalize this result for MO sequence spaces. More precisely we characterize the relatively weakly compact sets of a MO sequence space ℓ_Φ, which dual ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis.

In the second part of this note we prove that MO spaces ℓ_Φ with stabilized asymptotic ℓ_∞ dual are saturated with asymptotically isometric copies of ℓ_1. The notion of asymptotically isometric copy of ℓ_1 in a Banach space appeared in [7] and is used to investigate the fpp for non–expansive mappings of the non–reflexive subspaces of $L_1[0,1]$. Using the ideas of [1], [7] and [17] we show that any subspace of ℓ_Φ contains an asymptotically isometric copy of ℓ_1, provided that ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis and as a consequence of [7] this class of MO sequence spaces fails the fpp for closed, bounded, convex sets in ℓ_Φ and non–expansive maps on them. Let us mention that such a conclusion could have been drawn directly by using the recent characterization of the MO sequence spaces ℓ_Φ having fpp given in [16]: An MO sequence space has fpp for closed bounded convex sets and non–expansive maps on them iff it is reflexive. The examples at the end show that sometimes

[1]Research is partially supported by National Fund for Scientific Research of the Bulgarian Ministry of Education and Science, Contract MM-1401/04.

to check reflexivity is more difficult than to check that ℓ^*_Φ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis, due to the engagement of several constants in the definition of the δ_2–condition for a MO function Φ.

2 Preliminaries

We use the standard Banach space terminology from [11]. Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^\infty$ of Orlicz functions is called a Musielak–Orlicz function or MO function in short.

The MO sequence space ℓ_Φ, generated by a MO function Φ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg's norm in ℓ_Φ is defined by

$$\|x\|_\Phi = \inf\left\{r > 0 : \sum_{i=1}^\infty \Phi_i(x_i/r) \le 1\right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x = \{x_i\}_{i=1}^\infty \in \ell_\Phi$, such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for every $\lambda > 0$.

If the MO function Φ consists of one and the same function M one obtains the Orlicz sequence spaces ℓ_M and h_M.

Let $1 \le p_i$, $i \in \mathbb{N}$ be a sequence of reals. The MO sequence space ℓ_Φ, where $\Phi = \{t^{p_i}\}_{i=1}^\infty$ is called Nakano sequence space and is denoted by $\ell_{\{p_i\}}$. In [4] it was proved that two Nakano sequence spaces $\ell_{\{p_i\}}$, $\ell_{\{q_i\}}$ are isomorphic iff there exists $0 < C < 1$ such that

$$\sum_{i=1}^\infty C^{1/|p_i - q_i|} < \infty\,.$$

An extensive study of Orlicz and MO spaces can be found in [11] and [15] .

Definition 2.1 *We say that the MO function Φ satisfies the δ_2 condition at zero if there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$ such that for every $n \in \mathbb{N}$*

$$\Phi_n(2t) \le K\Phi_n(t) + c_n$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide iff Φ has δ_2 condition at zero.

Recall that given MO functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ, that is there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \le \Psi_n(t) + c_n \quad\text{and}\quad \Psi_n(Kt) \le \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$, [9] and [12].

Throughout this paper M will always denote Orlicz function while Φ - an MO function. As the properties we are dealing with are preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non–degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i^{'}(0) = 0$ and $\Phi_i(1) = 1$ [17].

Definition 2.2 *For an Orlicz function M, such that $\lim_{t\to 0} M(t)/t = 0$ the function*

$$N(x) = \sup\{t|x| - M(t) : t \geq 0\},$$

is called function complementary to M.

Definition 2.3 *The MO function $\Psi = \{\Psi_j\}_{j=1}^{\infty}$, defined by*

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geq 0\}\,, j = 1, 2, ..., n, ...$$

is called complementary to Φ.

Let us note that the condition $\lim_{t\to 0} M(t)/t = 0$ secures that the complementary function N is always non-degenerate. Observe that if N is function complementary to M, then M is complementary to N and if the MO function Ψ is complementary to the MO function Φ, then Φ is function complementary to Ψ. Throughout this paper the function complementary to the MO function Φ is denoted by Ψ.

It is well known that $h_M^* \cong \ell_N$ and $h_\Phi^* \cong \ell_\Psi$. Well known equivalent norm in ℓ_Φ is the Orlicz norm $\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^{\infty} x_j y_j : \sum_{j=1}^{\infty} \Psi_j(y_j) \leq 1\right\}$, which satisfies the inequalities (see e.g.[10])

$$\|\cdot\|_\Phi \leq \|\cdot\|_\Phi^O \leq 2\|\cdot\|_\Phi\,.$$

We will use the Hölder's inequality: $\sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^{\infty} \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^{\infty} \in \ell_\Psi$, where Φ and Ψ are complementary MO functions.

By $\{e_j\}_{j=1}^{\infty}$ and $\{e_j^*\}_{j=1}^{\infty}$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^{\infty}$ and element $x \in X$, $x = \sum_{i=1}^{\infty} x_i v_i$ we define $\mathrm{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$. We write $n \leq x$ if $n \leq \min\{\mathrm{supp} x\}$ and $x < y$ if $\max\{\mathrm{supp} x\} < \min\{\mathrm{supp} y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^{\infty}$ if $x = \sum_{i=p}^{q} x_i v_i$ for some finite p and q and we say that x is a normalized block vector if it is a block vector and $\|x\| = 1$.

Definition 2.4 *A Banach space X is said to be stabilized asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^{\infty}$, if there exists a constant $C \geq 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \leq x_1 < \ldots < x_n$ are successive normalized block vectors, then $\{x_i\}_{i=1}^{n}$ are C–equivalent to the unit vector basis of ℓ_∞^n, i.e.*

$$\frac{1}{C} \max_{1\leq i\leq n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq C \max_{1\leq i\leq n} |a_i|.$$

The following characterization of the stabilized asymptotic ℓ_∞ MO sequence spaces is due to Dew:

Proposition 2.1 *(Proposition 4.5.1 [5]) Let* $\Phi = \{\Phi_j\}_{j=1}^\infty$ *be a MO function. Then the following are equivalent:*
(i) h_Φ *is stabilized asymptotic* ℓ_∞ *(with respect to its natural basis* $\{e_j\}_{j=1}^\infty$*);*
(ii) there exists $\lambda > 1$ *such that for all* $n \in \mathbb{N}$*, there exists* $N \in \mathbb{N}$ *such that whenever* $N \leq p \leq q$ *and* $\sum_{j=p}^q \Phi_j(a_j) \leq 1$*, then*

$$\sum_{j=p}^{q} \Phi_j(a_j/\lambda) \leq \frac{1}{n}.$$

An easy sufficient condition for h_Φ to be stabilized asymptotic ℓ_∞ with respect to the unit vector basis is the following:

Proposition 2.2 *(Proposition 4.5.3 [5]) Let* $\varphi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}$*. If* $\lim_{j\to\infty} \varphi_\lambda(j) = \infty$ *for some* $\lambda > 1$ *then* h_Φ *is stabilized asymptotic* ℓ_∞ *with respect to the unit vector basis.*

Let X be a Banach space. By $Y \hookrightarrow X$ we denote that Y is isomorphic to a subspace of X.

Definition 2.5 *We say that a collection* $K \subset h_\Phi$ *has equi–absolutely continuous norms if for every* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that* $\sup\{\|\sum_{k=n}^\infty x_k e_k\| : x = \{x_k\}_{k=1}^\infty \in K\} < \varepsilon$ *for every* $n \geq N$*.*

Definition 2.6 *We say that a Banach space* $(X, \|\cdot\|)$ *is asymptotically isometric to* ℓ_1 *if it has a normalized basis* $\{v_n\}_{n=1}^\infty$ *such that for some sequence* $\{\lambda_n\}_{n=1}^\infty$ *increasing to* 1 *we have that*

$$\sum_{n=1}^{\infty} \lambda_n |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n v_n \right\| \tag{1}$$

for all $x = \sum_{n=1}^\infty t_n v_n \in X$*.*

Whenever $(X, \|\cdot\|)$ *contains a normalized sequence* $\{x^{(n)}\}_{n=1}^\infty$ *satisfying (1) then the closed linear span of* $\{x^{(n)}\}_{n=1}^\infty$ *is asymptotically isometric to* ℓ_1

We say that X is saturated with subspaces with the property (*) if in every infinite dimensional subspace Z of X there is an infinite dimensional subspace Y of Z isomorphic to a space with the property (*).

3 Weakly Compact Sets of MO Sequence Spaces

Lemma 3.1 *Let Φ be a MO function, which has δ_2 condition at zero and $K \subset h_\Phi$. Suppose that K fails to have equi–absolutely continuous norms. Then there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^\infty \subset K$, $\{p_n, q_n\}_{n=1}^\infty$, $p_n, q_n \in \mathbb{N}$, $p_n \le q_n < p_{n+1}$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ such that*

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0 \tag{2}$$

for every $n \in \mathbb{N}$.

Proof: Since K does not have equi–absolutely continuous norms there are $\varepsilon > 0$, $\{\alpha_n\}_{n\in\mathbb{N}}$, $\alpha_n \in \mathbb{N}$ and $\{z^{(n)}\} \subset K$ such that

$$\left\| \sum_{i=\alpha_n}^{\infty} z_i^{(n)} e_i \right\| > \varepsilon.$$

Let $n_1 = 1$. We choose $n_2 > n_1$ such that

$$\left\| \sum_{i=\alpha_{n_1}}^{\alpha_{n_2}-1} z_i^{(n_1)} e_i \right\| > \varepsilon/2.$$

Put $p_1 = \alpha_{n_1}$, $q_1 = \alpha_{n_2} - 1$, $x^{(1)} = z^{(n_1)}$. We choose $n_3 > n_2$ such that

$$\left\| \sum_{i=\alpha_{n_2}}^{\alpha_{n_3}-1} z_i^{(n_2)} e_i \right\| > \varepsilon/2.$$

Put $p_2 = \alpha_{n_2}$, $q_2 = \alpha_{n_3} - 1$, $x^{(2)} = z^{(n_2)}$.

If we have selected $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ by $x^{(s)} = z^{(n_s)}$, $p_s = \alpha_{n_s}$, $q_s = \alpha_{n_{s+1}} - 1$ for $1 \le s \le k$, then we choose $n_{k+1} > n_k$ such that

$$\left\| \sum_{i=\alpha_{n_{k+1}}}^{\alpha_{n_{k+2}}-1} z_i^{(n_{k+1})} e_i \right\| > \varepsilon/2.$$

Now we put $p_{k+1} = \alpha_{n_{k+1}}$, $q_{k+1} = \alpha_{n_{k+2}} - 1$ $x^{(k+1)} = z^{(n_{k+1})}$.

Obviously the sequence $\{x^{(k)}\}_{k=1}^\infty$ verifies (2) with $\varepsilon_0 = \varepsilon/2$. □

Lemma 3.2 *([2]) Let X be a Banach space. Suppose that $\{x_n\} \subset X$ is weakly null and $\{x_n^*\} \subset X^*$ is weakly* null. Then for each $\varepsilon > 0$ there is a subsequence $\{n_k\}_{k=1}^\infty$ of positive integers so that for each $k \in \mathbb{N}$ holds:*

$$\sum_{j \ne k} \left| x_{n_j}^*(x_{n_k}) \right| < \varepsilon.$$

Theorem 1 *Let Φ be a MO function, which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then any weakly null sequence in ℓ_Φ has equi–absolutely continuous norms.*

Proof: Theorem 1 Suppose the contrary. There is a weakly null sequence $\{x^{(n)}\}_{n=1}^\infty \subset \ell_\Phi$ that fails to have equi–absolutely continuous norms. By Lemma 3.1 there exist $\varepsilon_0 > 0$ and strongly increasing sequences $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$, $p_n, q_n \in \mathbb{N}$, $p_n \leq q_n < p_{n+1}$ such that

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0.$$

Choose $y^{(n)} \in h_\Psi$ such that supp $y^{(n)} = \{i\}_{i=p_n}^{q_n}$, $\sum_{k=p_n}^{q_n} \Psi_k(y_k^{(n)}) \leq 1$ and $\left| \sum_{k=p_n}^{q_n} y_k^{(n)} x_k^{(n)} \right| > \frac{3}{4}\varepsilon_0$. For a fixed $x \in \ell_\Phi$ by Holder's Inequality:

$$\left| \sum_{k=1}^{\infty} x_k y_k^{(n)} \right| = \left| \sum_{k=p_n}^{q_n} x_k y_k^{(n)} \right| \leq \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi \left\| y^{(n)} \right\|_\Psi^O .$$

As x is fixed and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ it follows that

$$\lim_{n\to\infty} \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi = 0.$$

Thus $\{y^{(n)}\}_{n=1}^\infty$ is weak* null sequence. By Lemma 3.2 there is a subsequence of naturals $\{n_k\}_{k=1}^\infty$ so that

$$\sum_{j\neq k} \left| \sum_{i=p_{n_j}}^{q_{n_j}} y_i^{(n_j)} x_i^{(n_k)} \right| < \varepsilon_0/2.$$

We claim that

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = \lim_{k\to\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = 0, \tag{3}$$

where $\lambda > 1$ is the constant from Proposition 2.1. Indeed, by assumption h_Ψ is stabilized asymptotic ℓ_∞ space and there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so that whenever $\sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j(y_j^{(n_k)}) \leq 1$ then the inequality $\sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \leq 1/m$ holds for every $q_{n_k} \geq p_{n_k} \geq N$. Thus $\lim_{n_k\to\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = 0$.

Therefore there is subsequence $\{n_{k_m}\}_{m=1}^{\infty}$ such that

$$\sum_{m=1}^{\infty}\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}} \Psi_i\left(\frac{y_i^{(n_{k_m})}}{\lambda}\right) \leq 1.$$

Let $y = \sum_{m=1}^{\infty} y^{(n_{k_m})}$. Obviously $y \in h_\Psi$ and since $\{x^{(n)}\}_{n=1}^{\infty}$ is weakly null we must have

$$\lim_{m\to\infty} y(x^{(n_{k_m})}) = \lim_{m\to\infty}\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})} = 0.$$

But

$$\left|\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})}\right| \geq \left|\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}} y_i^{(n_{k_m})} x_i^{(n_{k_m})}\right| - \sum_{j\neq m}\left|\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})}\right| \geq \frac{3}{4}\varepsilon_0 - \frac{1}{2}\varepsilon_0 = \frac{1}{4}\varepsilon_0,$$

a contradiction. □

Let us recall that C is weakly sequentially compact if every sequence of points in C has a subsequence weakly convergent to a point of C.

For the proof of the next result we need:

Theorem 2 *(Eberlein–Smulian, see e.g. [8]) Let X be a separable Banach space and C be a weakly closed subset of X. Then C is weakly compact if and only if C is weakly sequentially compact.*

By Theorem 1 it follows immediately

Corollary 3.1 *Let Φ be a MO function, which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the basis $\{e_j^*\}_{j=1}^{\infty}$. Then a bounded set $K \subset \ell_\Phi$ is relatively weakly compact iff K has equi–absolutely continuous norm.*

Proof: Necessity) Suppose that $K \subset h_\Phi$ is relatively weakly compact. If K fails to have equi–absolutely continuous norms then by Lemma 3.1 there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^{\infty} \subset K$, $\{p_n, q_n\}_{n=1}^{\infty}$, $p_n, q_n \in \mathbb{N}$, $p_n \leq q_n < p_{n+1}$ such that

$$\left\|\sum_{i=p_n}^{q_n} x_i^{(n)} e_i\right\| > \varepsilon_0$$

for every $n \in \mathbb{N}$.

By Eberlein–Smulian theorem there are $x \in \ell_\Phi$ and a subsequence $\{x^{(n_k)}\}_{n=1}^\infty$ such that $x^{(n_k)} \longrightarrow x$ weakly in ℓ_Φ. Thus by Theorem 1 $\{x^{(n_k)} - x\}_{k=1}^\infty$ has equi–absolutely continuous norms. Hence $\lim_{k\to\infty} \left\|\sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i)e_i\right\| = 0$ and obviously $\lim_{k\to\infty} \left\|\sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i\right\| = 0$. But

$$\varepsilon_0 < \left\| \sum_{i=p_{n_k}}^{q_{n_k}} x_i^{(n_k)} e_i \right\| \le \left\| \sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i \right\| + \left\| \sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i) e_i \right\| \underset{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction.

Sufficiency) Let K be a bounded set with equi–absolutely continuous norms. Let $\{x^{(n)}\}_{n=1}^\infty$ be an arbitrary sequence of elements in K. Obviously there exists L such that $|x_k^{(n)}| \le L$ for every $n, k \in \mathbb{N}$. Thus there exists a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that $\lim_{i\to\infty} x_k^{(n_i)} = x_k$ for every $k \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $s \ge N$ and every $i \in \mathbb{N}$ the inequality holds $\left\|\sum_{k=s}^\infty x_k^{(n_i)} e_k\right\| < \varepsilon/3$. Fix $s \ge N$. There is $M \in \mathbb{N}$ such that for every $n_i, n_j \ge M$ and every $k = 1, 2, \ldots, s$ the inequality $|x_k^{(n_i)} - x_k^{(n_j)}| \le \frac{\varepsilon}{3s}$ holds. Thus we can write the inequalities:

$$\begin{aligned}
\|x^{(n_i)} - x^{(n_j)}\| &= \left\|\textstyle\sum_{k=1}^\infty x_k^{(n_i)} e_k - \sum_{k=1}^\infty x_k^{(n_j)} e_k\right\| \\
&\le \left\|\textstyle\sum_{k=1}^s x_k^{(n_i)} e_k - \sum_{k=1}^s x_k^{(n_j)} e_k\right\| + \left\|\textstyle\sum_{k=s+1}^\infty x_k^{(n_i)} e_k - \sum_{k=s+1}^\infty x_k^{(n_j)} e_k\right\| \\
&\le \left\|\textstyle\sum_{k=1}^s |x_k^{(n_i)} - x_k^{(n_j)}| e_k\right\| + \left\|\textstyle\sum_{k=s+1}^\infty x_k^{(n_i)} e_k\right\| + \left\|\textstyle\sum_{k=s+1}^\infty x_k^{(n_j)} e_k\right\| \\
&< \tfrac{\varepsilon}{3s} s + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Consequently $\{x^{(n_i)}\}_{n=1}^\infty$ is a Cauchy sequence and thus it is norm convergent to $x \in \ell_\Phi$ and thus it is weakly convergent. □

Remark: Let us mention that for the proof of the sufficiency in Corollary 3.1 we do not need that ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$.

4 Fixed Point Property for MO Sequence Spaces

The next Lemma is similar to that in [17], where it is shown that for every normalized block basis $\{x^{(n)}\}_{n=1}^\infty$ of the unit vector basis $\{e_j\}_{j=1}^\infty$ in ℓ_Φ contains a subsequence such that $[x^{(n_i)}]_{i=1}^\infty$ is isomorphic to ℓ_1.

Lemma 4.1 *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then every normalized block basis $\{x^{(n)}\}_{n=1}^\infty$ of the unit vector basis $\{e_j\}_{j=1}^\infty$ in ℓ_Φ contains a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that $[x^{(n_i)}]_{i=1}^\infty$ is asymptotically isometric to ℓ_1.*

Proof: Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a normalized block basis of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ, where $x^{(n)} = \sum_{j=m_n+1}^{m_{n+1}} x_j^{(n)} e_j$, $\{m_n\}_{n=1}^{\infty}$ strictly increasing sequence of naturals. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence, such that $\lim_{n\to\infty} \lambda_n = 1$. For every $n \in \mathbb{N}$ there exists $y^{(n)} = \sum_{j=1}^{\infty} y_j^{(n)} e_j^* \in h_\Psi$ such that

$$\sum_{j=1}^{\infty} \Psi_j(y_j^{(n)}) \leq 1 \quad \sum_{j=1}^{\infty} y_j^{(n)} x_j^{(n)} \geq \lambda_n.$$

WLOG we may assume that supp $y^{(n)} \equiv$ supp $x^{(n)}$.

For the sequence $\{y^{(n)}\}_{n=1}^{\infty}$ and the constant $\lambda > 1$ from Proposition 2.1 holds:

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) = \lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) = 0.$$

The proof is essentially the same as for (3).

Now passing to a subsequence we get a sequence $\{y^{(n_k)}\}_{k\in\mathbb{N}}$, $y^{(n_k)} = \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^*$ such that

$$\sum_{k=1}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} \Psi_j \left(\frac{y_j^{(n_k)}}{\lambda} \right) \leq 1.$$

Denote $y = \sum_{k=1}^{\infty} y^{(n_k)} = \sum_{k=1}^{\infty} \left(\sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right)$. Obviously $y \in \ell_\Psi$ and $\|y\|_\Psi \leq \lambda$. As

$$\lim_{s\to\infty} \left\| \sum_{k=s}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi = 0$$

there exists $s_0 \in \mathbb{N}$ such that

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \leq \frac{1}{2}.$$

Consequently

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi^O \leq 2 \left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \leq 1.$$

Denote $\overline{y} = \sum_{k=s_0}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} e_j^*$. Then $\|\overline{y}\|_\Psi^O \leq 1$. Now using Hölder's inequality for any sequence $\{t_n\}_{n=1}^{\infty}$, such that $\sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \in \ell_\Phi$ we get

$$\begin{aligned} \left\| \sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \right\|_\Phi &\geq \frac{1}{\|\overline{y}\|_\Psi^O} \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} |t_{k-s_0+1} y_j^{(n_k)} x_j^{(n_k)}| \\ &\geq \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} x_j^{(n_k)} \geq \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \lambda_k. \end{aligned}$$

□

Theorem 3 *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1.*

Proof: According to a well known result of Bessaga and Pelczinski [3] every infinite dimensional closed subspace Y of ℓ_Φ has a subspace Z isomorphic to a subspace of ℓ_Φ, generated by a normalized block basis of the unit vector basis of ℓ_Φ. Now to finish the proof it is enough to observe that by Lemma 4.1 the space Z contains an asymptotically isometric copy of ℓ_1. □

By using a result from [7] that states that a Banach spaces containing an asymptotically isometric copy of ℓ_1 fail the fixed point property for closed, bounded, convex sets and non–expansive (contractive) maps on them, we easily get:

Corollary 4.1 *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then ℓ_Φ fails the fixed point property for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.*

We give at the end some examples of MO sequence space, saturated with asymptotically isometric copies of ℓ_1.

Example 1:([17]) Sometimes we know only the complementary function Ψ. For example let the MO function $\Psi = \{\Psi_j\}_{j=1}^\infty$ be defined by $\Psi_j = e^{\alpha_j} e^{-\frac{\alpha_j}{|x|^{c_j}}}$, where $\lim_{j\to\infty} \alpha_j = \infty$ and $0 < c_j$. Then ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ because

$$\lim_{j\to\infty} \inf \left\{ \frac{\Psi_j(2x)}{\Psi_j(x)} : 0 \le x \le 1 \right\} = \lim_{j\to\infty} \inf \left\{ e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}|x|^{c_j}}} : 0 \le x \le 1 \right\} = \lim_{j\to\infty} e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}}} = \infty.$$

Thus we conclude that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

Example 2:([5]) Consider the Nakano sequence space $\ell_{\{p_n\}}$, where $p_n = \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)}$.

It is well known that $\ell^*_{\{p_n\}} \cong \ell_{\{q_n\}}$, where $1/p_n + 1/q_n = 1$, i.e. $q_n = \log_2(n+1)$. It is easy to see that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)} = 1$ and thus according to [4] and [12] $\ell_{\{p_n\}}$ is saturated with spaces isomorphic to ℓ_1. Moreover according to [5] $\ell_{\{q_n\}}$ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ and thus $\ell_{\{p_n\}}$ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

References

[1] J. Alexopoulos, On Subspaces of non-Reflexive Orlicz Spaces. *Quaestiones Mathematicae* **21** (3 and 4) (1998), 161-175.

[2] J. Alexopoulos, De La Vallee Poussins Theorem and Weakly Compact Sets in Orlicz Spaces. *Quaestiones Mathematicae* **17** (1994), 231-248.

[3] C. Bessaga, A. Pelczynski, On bases and unconditional convergence of series in Banach Spaces, *Studia Math.*, **17** (1958), 165–174.

[4] O. Blasco, P. Gregori, Type and Cotype in Nakano Sequence Spaces $\ell_{(\{p_n\})}$, preprint

[5] Neil Dew, Asymptotic sructure of banach spaces, PhD Thesis (St. John's College University of Oxford) 2002.

[6] J. Diestel A survey of results related to the Danford–Pettis property, Integration, topology and geometry in linear spaces, Proc. Conf. Chapel Hill, N.C. 1979, *Contemp. Math.*, **2** (1980), 15–60.

[7] P. Dowling, C. Lennard, Every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property *Proc. of the Amer. Math. Soc.*, **125** (1997), 443-446.

[8] P. Habala, P. Hájek, V. Zizler, Introduction to Banach spaces, *Charles University Press, Prague.* (1996).

[9] A. Kaminska, Indices, Convexity and Concavity in Musielak–Orlicz Spaces *Functiones et Approximatio*, **XXVI** (1998), 67-84.

[10] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general *Indagationes Mathematicae*, **11** (2000), 573–585.

[11] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I, Sequence spaces, Springer–Verlag, Berlin, 1977

[12] R. Maleev, B. Zlatanov, Smoothness in Musielak–Orlicz sequence spaces *Comptes renus de l'Académie bulgare des Sciences*, **55** (2002), 11-16.

[13] B. Maurey, V.D. Milman, N. Tomczak–Jaegermann, Asymptotic infinite–dimensional theory of Banach spaces *Geometric aspects of functional analysis (Israel 1992–1994)*, Operator Theory Advances and Applications **77** Birkhauser, (1995), 149-175.

[14] V.D. Milman, N. Tomczak–Jaegermann, Asymptotic ℓ_p spaces and bounded distortions. Banach spaces (Merida) *Contemporary Mathematics*, **144** American Mathematical Society (1992), 173-195.

[15] J. Musielak, Lecture Notes in Mathematics, **1034** Springer–Verlag, Berlin, 1983

[16] H. Bevan Thompson, Yunan Cui, The fixed point property in Musielak–Orlicz sequence spaces. *Comment. Math. Univ. Carolinae*, **42** (2001), 299-309.

[17] B. Zlatanov, Schur property and ℓ_p isomorphic copies in Musielak–Orlicz sequence spaces. *Bulletin of the Australian Math. Soc.* (to appear).

Department of Mathematics and Informatics
Plovdiv University,
24 “Tzar Assen” str.
Plovdiv, 4000
Bulgaria
e-mail:bobbyz@pu.acad.bg