AN ETUDE ON ONE SHARYGIN'S PROBLEM

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Abstract: By the methods of the synthetic geometry we investigate properties of objects generated from a complete quadrangle and a line, which lies in its plane. We start with a problem from the book of Sharygin "Problems in Plane Geometry". We generalize this problem with the help of Pappus, Desargues and Pascal's Theorems and we discover new concurrent lines, collinear points, and conic sections.

1. Introduction

The book of Sharygin "Problems in Plane Geometry" [5] is a collection of interesting and various in difficulty problems. A challenge that can unfold an entire world can be even a small mathematical problem. The Dynamic Geometry Software (DGS) facilitates substantially the efforts of the mathematicians in this direction [3], [4]. We would like to illustrate an evolution of the idea implemented in a small school problem ([5], Problem 34, p. 72) that is based on projective and combinatorial methods with the help of DGS.

2. Preliminary

The investigations in the present work relate the Euclidian model of the Projective plane, i.e. the Euclidian plane complimented with its infinite points and its infinite line ω.

Theorem 1. *(Pappus) Let be given two lines g and g' . If $A, B, C \in g$ and $A', B', C' \in g$, then the points $P = BC' \cap CB'$, $Q = AC' \cap CA'$, $R = AB' \cap BA'$ are collinear.*

A triangle is called the set of three noncollinear points and their three joining lines.

Theorem 2. *(Desargues) The connecting lines of the couples of corresponding vertices of two triangles ABC and $A'B'C'$ are intersecting at a point S if and only if the intersection points of the couples of corresponding sides $P = BC \cap BC$, $Q = AC \cap AC$, $R = AB \cap AB$ lie on a line s.*

Two triangles that satisfied the conditions of Theorem 2 are called perspective. The point S is called perspective center and the line s is called a perspective axis.

Theorem 3. *(Pascal) A hexagon $AB'CA'BC'$ is inscribed in a conic section k if and only if the points $P = AB' \cap A'B$, $Q = B'C \cap BC'$, $R = CA' \cap C'A$ are collinear.*

The line that is incident with the points P, Q, R is called Pascal's line.

A complete quadrangle is called the set of four points P, Q, R, S, of which no three are collinear, and the lines QR, PS, RP, QS, PQ, RS. The intersections of the opposite sides $A = QR \cap PS$, $B = RP \cap QS$, $C = PQ \cap RS$ are called diagonal points and they are the vertices of the diagonal triangle of the complete quadrangle $PQRS$.

Theorem 4. *(Pappus–Desargues) The three pairs of opposite sides of a complete quadrangle intersect a line, which is not incident with any of its vertices, in three pairs of corresponding points for one and the same involution.*

Theorem 5. *([2], Theorem 6.43, p. 73) The diagonal triangle of every inscribed in a conic section complete quadrangle is self–polar.*

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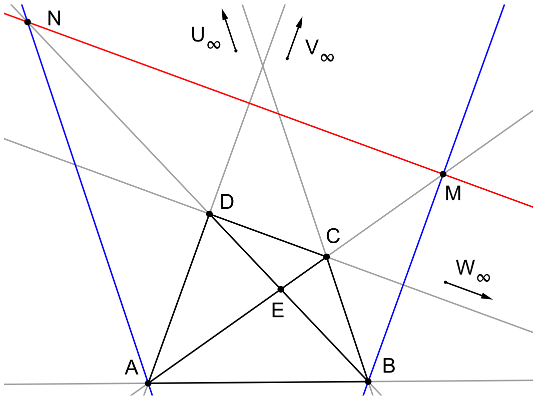

FIGURE 1. Sharygin's Problem

The present investigation is inspired by one problem of Sharygin. Our results can be considered as improvisations on the idea, that is encoded into the next problem.

Problem 1. *([5], problem 34, p. 72) Let $ABCD$ be a quadrangle and the points $M \in AC$ and $N \in BD$ be such that $BM \parallel AD$ and $AN \parallel BC$. Prove that $MN \parallel CD$.*

The proof given by Sharygin is based on the proportionality of the corresponding sides of the similar triangles EMB and EAD, EBC and ENA, where $E = AC \cap BD$ and Thales' Theorem.

3. A Generalized Sharygin's Problem

We would like to present another solution of Problem 1, that is based on Pappus' Theorem. This solution will help us to investigate some properties of a quadrangle and a line which lies in its plane. We would like to mention that a similar approach have been used for generalizing of another Sharygin's Problem in ([3], Problems 12 and 13).

Proof. Let consider the quadrangle $ABCD$ in the extended Euclidian plane and let us introduce the notations: $U_\infty = BC \cap \omega$, $V_\infty = AD \cap \omega$, $W_\infty = CD \cap \omega$ (Fig. 1). Let us consider the ordered triads of collinear points (B, C, U_∞) and (A, V_∞, D). According to Theorem 1 it follows that the points $W_\infty = CD \cap U_\infty V_\infty = CD \cap \omega$, $M = BV_\infty \cap AC$ and $N = AU_\infty \cap BD$ are collinear, which means that $W_\infty \in MN$, i.e. $MN \parallel CD$. □

The proof, which we have presented is not only shorter, but it gives a possibility to develop the idea encoded into this small school Sharygin's problem. We can generate 48 similar problems band together with a common simple solution. We will state Problem 1 in the language of the extended Euclidian plane.

Problem 1* Let $ABCD$ be a quadrangle and U_∞ be the infinite point on the line BC and V_∞ be the infinite point on the line AD. Prove that the points $M = BV_\infty \cap AC$, $N = AU_\infty \cap BD$ and $W_\infty = CD \cap \omega$ are collinear, which means that $MN \parallel CD$.

We notice in the proof of Problem 1, that the infinite points U_∞ and V_∞ can be replaced with finite points $U \in BC$ or $V \in AD$. Thus we can state a variant of Problem 1*.

Problem 2. *Let $ABCD$ be a quadrangle and $U \in BC$ and $V \in AD$ be arbitrary points. Prove that the points $M = BV \cap AC$, $N = AU \cap BD$ and $W = CD \cap UV$ are collinear, i.e. the lines MN, CD, UV are concurrent.*

Proof. It is enough to apply Theorem 1 for the ordered triads of points (B, C, U) and (A, V, D) (Fig. 2, Left) . □

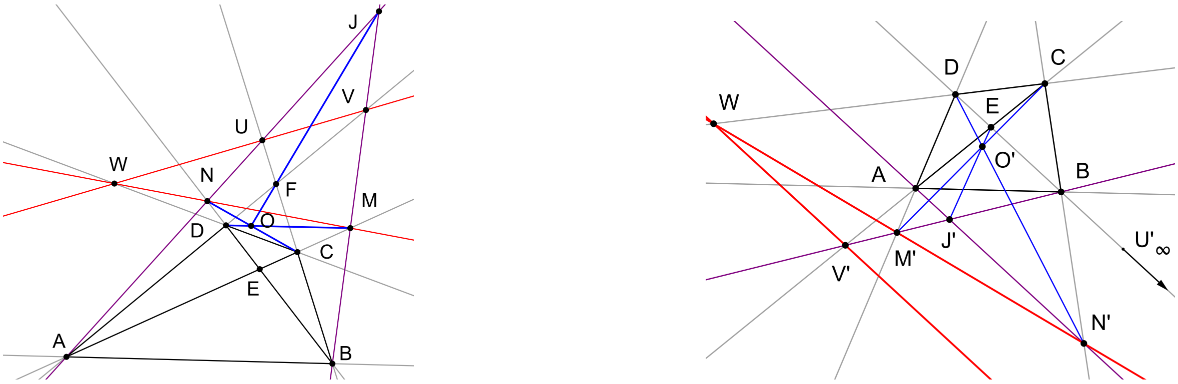

FIGURE 2. *Left* Sharygin's Problem for finite points U and V. *Right* Sharygin's Problem for one finite point V and one infinite point U_∞

Since there exists two possibilities for the points U and V – to be finite or infinite, then their possible combinations are four. We observe that the role of pair of lines (BC, AD) and (AC, BD) can be interchanged in Problem 1 and Problem 1*. Let us state another variant of Sharygin's Problem.

Problem 3. *Let $ABCD$ be a quadrangle and $V' \in AC$ be an arbitrary point and U'_∞ is the infinite point on the line BD. Let us denote $M' = BV' \cap AD$, $N' = AU'_\infty \cap BC$. Prove that the lines CD, $V'U'_\infty$ and $M'N'$ are concurrent (Fig. 2, Right).*

Proof. It is enough to apply Theorem 1 for the ordered triads of points (A, C, V') and (B, U'_∞, D). □

The Dynamic Geometry Softwares facilitate the research work. They can help us to state a hypotheses, which are necessary to be proven after that with synthetic methods or with ACS [4]. Thus with the notations in Fig. 2 to the left we can state a hypothesis: *The lines DM, CN, FJ are concurrent.* Using the notation in Fig. 2 to the right we can state the hypothesis: *The lines DN', CM', EJ' are concurrent.* We will prove the first statement. Let us consider the triangles DCF and MNJ. From Problem 2 the points $W = DC \cap MN$, $U = CF \cap NJ$ and $V = DF \cap MJ$ are collinear. According to Theorem 2 the triangles DCF and MNJ are perspective ones with a perspective axis UV. Therefore the connecting lines of the pairs of corresponding vertices i.e. DM, CN, FJ are concurrent.

We will replace the quadrangle $ABCD$ with the complete quadrangle $A_1A_2A_3A_4$ in order to be able to state all of the cases in one problem.

Problem 4. *(Generalized Sharygin's problem) Let $A_1A_2A_3A_4$ be a complete quadrangle, A_i, A_j be arbitrary pair of its vertices and the points $U_{js} \in A_jA_s$ and $U_{ik} \in A_iA_k$ be arbitrary chosen, where $i, j, k, s \in \{1, 2, 3, 4\}$ and any two of them are different. Let us denote $g = U_{js}U_{ik}$, $I = A_iA_k \cap A_jA_s$, $M = A_iU_{js} \cap A_jA_k$, $N = A_jU_{ik} \cap A_iA_s$ and $J = A_iM \cap A_jN$. Prove that:*
1) The lines g, MN, A_kA_s are concurrent;
2) The lines A_sM, A_kN, IJ are concurrent.

Let us point out that the points U_{is} and U_{si} coincide for any $i, s \in \{1, 2, 3, 4\}$, $i \neq s$. Therefore we will use the notation U_{is}, $i < s$ for all the figures and examples.

Proof. 1) We apply Theorem 1 to the ordered triads of collinear points (A_i, A_k, U_{ik}) and (A_j, U_{js}, A_s) and we get that the points M, N and $W = g \cap A_kA_s$ are collinear, i.e. the lines g, MN, A_kA_s are concurrent.

2) Let us consider the triangles A_sA_kI and MNJ. We establish that $A_sI \cap MJ = A_sA_j \cap A_iU_{js} = U_{js} \in g$ and $A_kI \cap NJ = A_kA_i \cap NA_j = A_kA_i \cap A_jU_{ik} = U_{ik} \in g$. From 1) we have $A_kA_s \cap NM = W \in g$. Consequently the triangles A_sA_kI and MNJ are perspective with a perspective axis g. According to Theorem 2 the lines A_sM, A_kN, IJ are concurrent. □

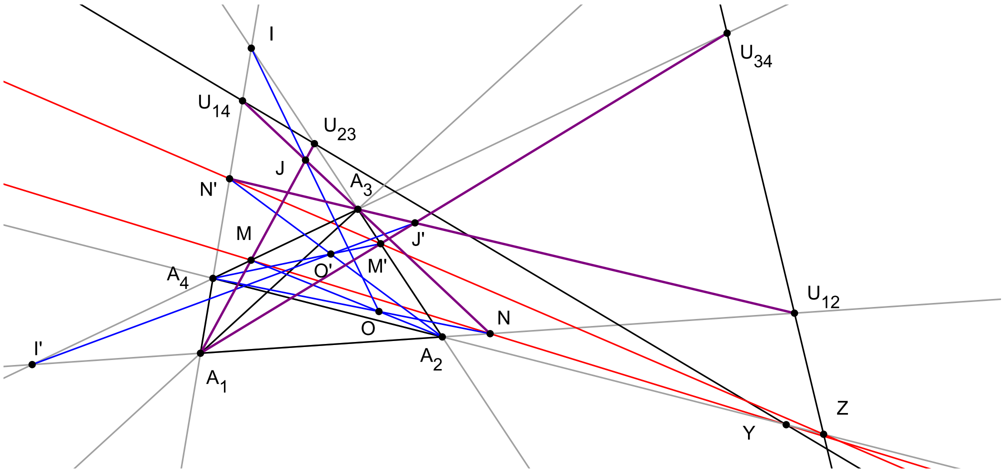

FIGURE 3. The four Sharygin's points for $i = 1$, $j = 3$

Definition 1. *If $A_1A_2A_3A_4$ is a complete quadrangle and the points $U_{js} \in A_jA_s$ and $U_{ik} \in A_iA_k$ are arbitrary chosen, then the points $M = A_iU_{js} \cap A_jA_k$ and $N = A_jU_{ik} \cap A_iA_s$, will be called Sharygin's points for $A_1A_2A_3A_4$, which are associate with the line $g = U_{js}U_{ik}$ and the vertices A_i, A_j.*

There are $C_4^2 = 6$ possible choices for the pairs of vertices (A_i, A_j). For any chosen pair (A_i, A_j) there are two different problems, because of the existence of two mutually exclusive possibilities for the indices s and k. The combinations of the points U_{js} and U_{ik} are four, because any one can be finite or infinite. Therefore the number of the specific tasks that can be formulated from Problem 4 is $6.2.4 = 48$.
For example Problem 1 is a particular case of the problem 4 for $i = 1$, $j = 2$, $s = 3$, $k = 4$, where U_{23} and U_{14} are the infinite points of the lines A_2A_3 and A_1A_4, respectively.

The visualization of the particular cases of Problem 4 can be done easily with the help of the special function "Swap finite and infinite points" in DGS – Sam [3]. It is enough to sketch the problem for one of four combinations of the points U_{js} and U_{ik} and the other three are obtained with the help of the function "Swap finite and infinite points".

For the convenience of the reader we will state a particular case of Problem 4 for $i = 1$, $j = 3$ in the next problem. There are two choices for the indices s and k: $s = 2$, $k = 4$ or $s = 4$, $k = 2$. That is why one can see four different Sharygin's points, associated with the vertices A_1 and A_3.
Problem 4* Let $A_1A_2A_3A_4$ be a complete quadrangle, A_1, A_3 be a pair of its vertices. Let the finite points $U_{23} \in A_2A_3$, $U_{14} \in A_1A_4$, $U_{34} \in A_3A_4$, $U_{12} \in A_1A_2$ be arbitrary chosen. Let us denote $I = A_1A_4 \cap A_3A_2$, $M = A_1U_{23} \cap A_3A_4$, $N = A_3U_{14} \cap A_1A_2$, $J = A_1M \cap A_3N$, $M' = A_1U_{34} \cap A_3A_2$, $N' = A_3U_{12} \cap A_1A_4$, $J' = A_1M' \cap A_3N'$ (Fig. 3) . Prove that:
1) The lines $U_{23}U_{14}$, MN, A_2A_4 are concurrent; the lines $U_{34}U_{12}$, $M'N'$, A_2A_4 are concurrent;
2) The lines A_2M, A_4N, IJ are concurrent; the lines A_2N', A_4M', $I'J'$ are concurrent.

4. Sharygin's Conic Sections

We will put an additional condition on the points $U_{ij} \in A_iA_j$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ to be collinear for the next investigations.

Definition 2. *Let $A_1A_2A_3A_4$ be a complete quadrangle and let g be a line, which lies in the plane of $A_1A_2A_3A_4$ and it is not incident with any vertex of $A_1A_2A_3A_4$ or with any of its diagonal points. We call the pair $(A_1A_2A_3A_4, g)$ a (q, l)–pair.*

For any (q,l)–pair $(A_1A_2A_3A_4, g)$ we will use in the sequel the notation

$$U_{ij} = U_{ji} = g \cap A_iA_j, i, j \in \{1,2,3,4\}, i \neq j. \tag{1}$$

According to Theorem 4 the pairs of points (U_{12}, U_{34}), (U_{13}, U_{24}) and (U_{14}, U_{23}) are pairs of conjugated points for one and the same involution φ. The additional condition imposed on the line g not to be incident with a diagonal point excludes the possibility the Sharygin's points to coincide with vertices of $A_1A_2A_3A_4$.

When the points U_{ij} are collinear it will be easier to introduce a unified notation for the Sharygin's points. The two Sharygin's points associated with the vertices A_i, A_j and the points U_{js}, U_{ik} and the two Sharygin's points associated with the vertices A_i, A_j and the points U_{jk}, U_{is} are determined by the pair of vertices A_i, A_j and the line g. Thus we can denote the four Sharygin's points associated with the pair of vertices A_i, A_j of the (q,l)–pair $(A_1A_2A_3A_4, g)$ with

$$\begin{aligned} M_{ij}^k &= A_iU_{js} \cap A_jA_k,\ M_{ji}^s = A_jU_{ik} \cap A_iA_s,\\ M_{ij}^s &= A_iU_{jk} \cap A_jA_s,\ M_{ji}^k = A_jU_{is} \cap A_iA_k. \end{aligned} \tag{2}$$

Four Sharygin's points are connected to any pair of vertices for any (q,l)–pair $(A_1A_2A_3A_4, g)$ and they are the intersecting points of the connecting lines of the vertices A_i and A_j with the pair of conjugated points for the Pappus–Desargues involution, which is induced by $A_1A_2A_3A_4$ into g, with the third pair of opposite sides. Let us introduce the points

$$\begin{aligned} I &= A_iA_k \cap A_jA_s, \quad I' = A_iA_s \cap A_jA_k;\\ J &= A_iU_{js} \cap A_jU_{ik},\ J' = A_iU_{jk} \cap A_jU_{is};\\ L &= A_kU_{js} \cap A_sU_{ik},\ L' = A_sU_{jk} \cap A_kU_{is}, \end{aligned} \tag{3}$$

which we will need in the sequel.

Theorem 6. *Let $(A_1A_2A_3A_4, g)$ be a (q,l)–pair. Then the following hold true:*
1) The lines $M_{ij}^kM_{ji}^s$, $M_{ij}^sM_{ji}^k$, are incident with the point U_{ks};
The lines $M_{sk}^jM_{ks}^i$, $M_{sk}^iM_{ks}^j$, are incident with the point U_{ij}.
2) The lines $A_iM_{ks}^j, A_jM_{sk}^i, M_{ij}^kA_s, M_{ji}^sA_k, JI, I'L'$ are concurrent;
The lines $A_iM_{sk}^j, A_jM_{ks}^i, M_{ji}^kA_s, M_{ij}^sA_k, IL, J'I'$ are concurrent.
3) The lines $M_{ij}^kM_{ij}^s$, $M_{ji}^sM_{ji}^k$, A_iA_j, IJ, $I'J'$ are concurrent;
The lines $M_{sk}^jM_{sk}^i$, $M_{ks}^iM_{ks}^j$, A_sA_k, IL, $I'L'$, are concurrent.

Proof. Just for a convenience of the reader we will write and the Sharygin's points associated with the pair of vertices A_s, A_k:

$$\begin{aligned} M_{sk}^j &= A_sU_{ik} \cap A_jA_k, \quad M_{ks}^i = A_kU_{js} \cap A_iA_s,\\ M_{sk}^i &= A_sU_{jk} \cap A_iA_k, \quad M_{ks}^j = A_kU_{is} \cap A_jA_s. \end{aligned} \tag{4}$$

1) We apply Theorem 1 for the ordered triads of points (A_i, A_k, U_{ik}) and (A_j, U_{js}, A_s) and using (3) we get that the points M_{ij}^k, M_{ji}^s and $A_kA_s \cap U_{ik}U_{js}$ are collinear. Therefore the lines $M_{ij}^kM_{ji}^s$, A_kA_s and $g = U_{ik}U_{js}$ are incident with the point U_{ks}. Applying Theorem 1 once more for the ordered triads of points (A_i, A_s, U_{is}) and (A_j, U_{jk}, A_k) and using (3) we get that the points M_{ij}^s, M_{ji}^k and $A_kA_s \cap U_{is}U_{jk}$ are collinear. Therefore the lines $M_{ij}^sM_{ji}^k$, A_kA_s and g are incident with the point U_{ks}. With the help of (4) and Theorem 1, applied consequently for the ordered triads of points (A_s, A_j, U_{sj}) and (A_k, U_{ik}, A_i), (A_s, A_i, U_{si}) and (A_k, U_{jk}, A_j), we prove that the lines $M_{sk}^jM_{ks}^i$ and $M_{sk}^iM_{ks}^j$ are incident with the point U_{ij}.

2) Let us consider the triangles A_sA_kI and $M_{ij}^kM_{ji}^sJ$. Taking into account (1), (2), (3) and 1) in Theorem 6 we get that the points $A_sA_k \cap M_{ij}^kM_{ji}^s = U_{ks}$, $A_kI \cap M_{ji}^sJ = A_iA_k \cap A_jU_{ik} = U_{ik}$ and

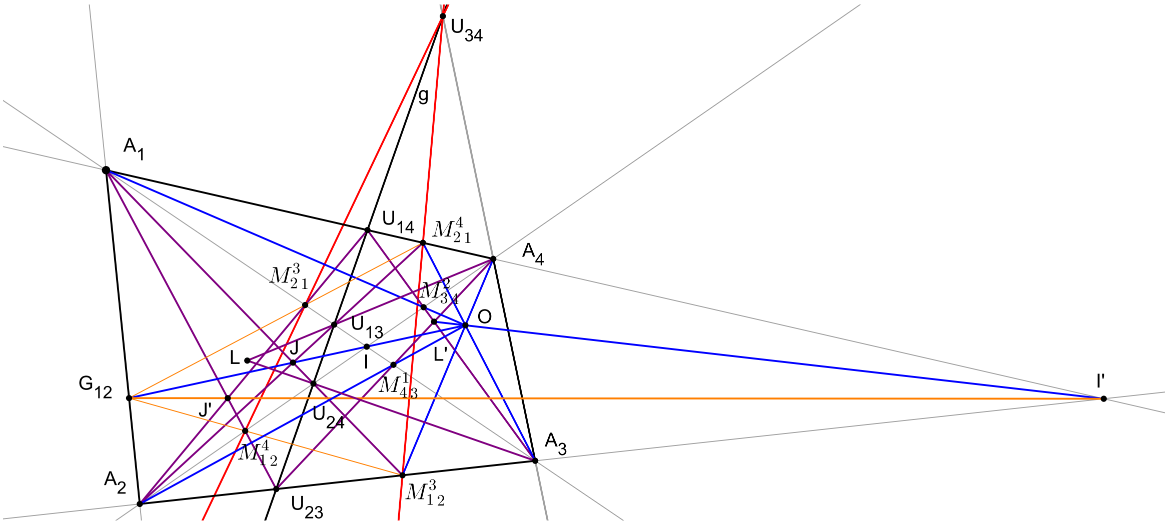

FIGURE 4. Theorem 6 for $i = 1$, $j = 2$, $k = 3$, $s = 4$

$A_sI \cap M^k_{ij}J = A_jA_s \cap A_iU_{js} = U_{js}$ are collinear. Then from Theorem 2 it follows that the lines $A_sM^k_{ij}$, $A_kM^s_{ji}$ and IJ are concurrent and let's denote $O = A_sM^k_{ij} \cap A_kM^s_{ji} \cap IJ$. Taking into account (1), (2), (3) and 1) in Theorem 6 we get again that the triangles A_sA_kL' and $M^k_{ij}M^s_{ji}I'$ are perspective with a perspective axis g. Therefore these triangles have a perspective center $O^a = A_sM^k_{ij} \cap A_kM^s_{ji} \cap L'I'$. It is easy to see that $O = O^a$. Using (1), (2), (3), (4) and 1) in Theorem 6 we establish that the pairs of triangles A_iA_jJ and $M^j_{ks}M^i_{sk}I$, A_iA_jI' and $M^j_{ks}M^i_{sk}L'$ are perspective with a perspective axis g. Hence $A_iM^j_{ks} \cap A_jM^i_{sk} \cap JI = O^b$ and $A_iM^j_{ks} \cap A_jM^i_{sk} \cap I'L' = O^c$. It is easy to observe that $O^b = O^c$ and then $A_iM^j_{ks} \cap A_jM^i_{sk} \cap I'L' \cap JI = O^c$. Comparing all results we get $O = O^a = O^b = O^c$ or

$$A_iM^j_{ks} \cap A_jM^i_{sk} \cap A_sM^k_{ij} \cap A_kM^s_{ji} \cap JI \cap I'L' = O. \tag{5}$$

By similar considerations for the pairs of triangles A_kA_sI' and $M^s_{ij}M^k_{ji}J'$, A_kA_sL and $M^s_{ij}M^k_{ji}I$ we prove that they have a common perspective center $O' = A_kM^s_{ij} \cap A_sM^k_{ji} \cap I'J' \cap LI$. Repeating the considerations for the pairs of triangles A_iA_jI and $M^j_{sk}M^i_{ks}L$, A_iA_jJ' and $M^j_{sk}M^i_{ks}I'$ we get that they have a common perspective center $\hat{O} = A_iM^j_{sk} \cap A_jM^i_{ks} \cap IL \cap J'I'$. Comparing these results we can write

$$A_iM^j_{sk} \cap A_jM^i_{ks} \cap A_sM^k_{ji} \cap A_kM^s_{ij} \cap I'J' \cap LI = O'. \tag{6}$$

3) Let us consider the triangles $M^k_{ij}A_jJ$ and $M^s_{ij}A_iI$. Using (1), (2) and (3) we get that the intersecting points U_{jk}, U_{ik}, U_{js} of their corresponding sides lie on the line g. Therefore the triangles $M^k_{ij}A_jJ$ and $M^s_{ij}A_iI$ are perspective with a perspective axis g. According to Theorem 2 the lines $M^k_{ij}M^s_{ij}$, A_jA_i, JI are passing through a common point. Let us denote $M^k_{ij}M^s_{ij} \cap A_jA_i \cap JI = G_{ij}$.

After similar considerations for the triangles $M^s_{ji}A_iJ$ and $M^k_{ji}A_jI$ we prove that they are perspective with a perspective axis g. According to Theorem 2 they have a perspective center and we can write $M^s_{ji}M^k_{ji} \cap A_iA_j \cap JI = \widehat{G}_{ij}$. Therefore $G_{ij} = \widehat{G}_{ij}$.

At the end let us consider the triangles $M^s_{ij}A_jJ'$ and $M^k_{ij}A_iI'$. They are perspective again with a perspective axis g. Therefore the connecting lines $M^k_{ij}M^s_{ij}$, A_iA_j, $J'I'$ of their corresponding vertices are concurrent lines and it is easy to see they are passing through the point G_{ij}. Thus we received

$$M^k_{ij}M^s_{ij} \cap M^s_{ji}M^k_{ji} \cap A_iA_j \cap JI \cap J'I' = G_{ij}. \tag{7}$$

Now with the help of the pairs of triangles $M^j_{sk}A_sI'$ and $M^i_{sk}A_kL'$, $M^i_{ks}A_sI'$ and $M^j_{ks}A_kL'$, $M^i_{sk}A_sI$ and $M^j_{sk}A_kL$, (1), (2), (3) and (4) we obtain

$$M^j_{sk}M^i_{sk} \cap M^i_{ks}M^j_{ks} \cap A_sA_k \cap LI \cap L'I' = G_{sk}. \tag{8}$$

□

We present a particular case of Theorem 6 for $i = 1$, $j = 2$, $k = 3$, $s = 4$ on Fig. 4.

By Theorem 6 it follows that for any given (q, l)–pair $(A_1A_2A_3A_4, g)$ there exists one point $G_{ij} \in A_iA_j$ for every side A_iA_j with the properties (7).

Definition 3. *Let $(A_1A_2A_3A_4, g)$ be a (q, l)–pair. The intersection point of the line A_iA_j with the line $M^s_{ij}M^k_{ij}$ will be denoted with G_{ij} and will be called a G_{ij}–point for the pair $(A_1A_2A_3A_4, g)$, associated with the vertices A_i and A_j.*

Proposition 1. *The G_{ij}–point is a harmonic conjugate point of the point U_{ij} with respect to the pair A_i, A_j.*

Proof. Let us consider the complete quadrangle $M^s_{ij}M^k_{ij}U_{js}U_{jk}$ which vertices are defined by (1) and (2). The points $A_j = M^s_{ij}U_{js} \cap M^k_{ij}U_{jk}$ and $A_i = M^s_{ij}U_{jk} \cap M^k_{ij}U_{js}$ are diagonal points for $M^s_{ij}M^k_{ij}U_{js}U_{jk}$. From (1) and (7) it follows that the pair of opposite sides $M^s_{ij}M^k_{ij}$ and $U_{js}U_{jk}$, which are passing through the third diagonal point, intersect the side A_iA_j in the points G_{ij} and U_{ij}, respectively. So, following [1] we can write the harmonic group $H(G_{ij}U_{ij}, A_iA_j)$. □

Theorem 6 holds true also if the line g is replaced with the infinite line ω. In this case the points U_{ij} are infinite points wherefore the points G_{ij} are midpoints for the Euclidean segments A_iA_j. The new element in the proof of Theorem 6 will be a simplification in the proof of 3). Indeed, now the quadrangles A_iJA_jI, $A_iJ'A_jI'$, $A_iM^s_{ij}A_jM^k_{ij}$, $A_iM^s_{ji}A_jM^k_{ji}$ are parallelograms and it follows that the segments IJ, $I'J'$, $M^s_{ij}M^k_{ij}$, $M^s_{ji}M^k_{ji}$ and A_iA_j have a common midpoint G_{ij}. The case for $i = 1$ and $j = 3$ is presented in Fig. 5 to the right.

Proposition 2. *Any two vertices and the four Sharygin's points that are connected with them lie on a conic section.*

Proof. Let us consider the hexagon $A_iM^k_{ij}M^s_{ij}A_jM^s_{ji}M^k_{ji}$. From (2), (3) and 3) in Theorem 6 it follows that the points $A_iM^k_{ij} \cap A_jM^s_{ji} = A_iU_{js} \cap A_jU_{ik} = J$, $M^k_{ij}M^s_{ij} \cap M^s_{ji}M^k_{ji} = G_{ij}$ and $M^s_{ij}A_j \cap M^k_{ji}A_i = A_jA_s \cap A_iA_k = I$ are collinear. Then according to Theorem 3 the hexagon $A_iM^s_{ij}M^s_{ij}A_jM^s_{ji}M^k_{ji}$ is inscribed in a conic section. □

Definition 4. *The conic section, which passes through the vertices A_i and A_s of the complete quadrangle $A_1A_2A_3A_4$ from the (q, l)–pair $(A_1A_2A_3A_4, g)$ and through the four Sharygin's points M^j_{is}, M^k_{is}, M^j_{si}, M^k_{si} associated with these vertices will be called Sharygin's curve for the (q, l)–pair $(A_1A_2A_3A_4, g)$ and will be denoted with k_{is}.*

From Proposition 2 it follows that there are six Sharygin's curves for any (q, l)–pair $(A_1A_2A_3A_4, g)$. Fig. 5 to the left presents the Sharygin's curves k_{12} and k_{34}, when g is a finite line. Fig. 5 to the right presents the Sharygin's curve k_{12}, when g is the infinite line ω.

We have presented a technique for GeoGebra to simulate the special function "Swap finite and infinite points" of DGS – Sam in [4]. With the help of this technique Fig. 5 to the right, after removing the Sharygin's curve k_{34} just for simplicity of notations, can be generated from Fig. 5 to the left and vice versa.

Proposition 3. *The G_{ij} –point is the pole of the line g with respect to the Sharygin's curve k_{ij} for the (q, l)–pair $(A_1A_2A_3A_4, g)$.*

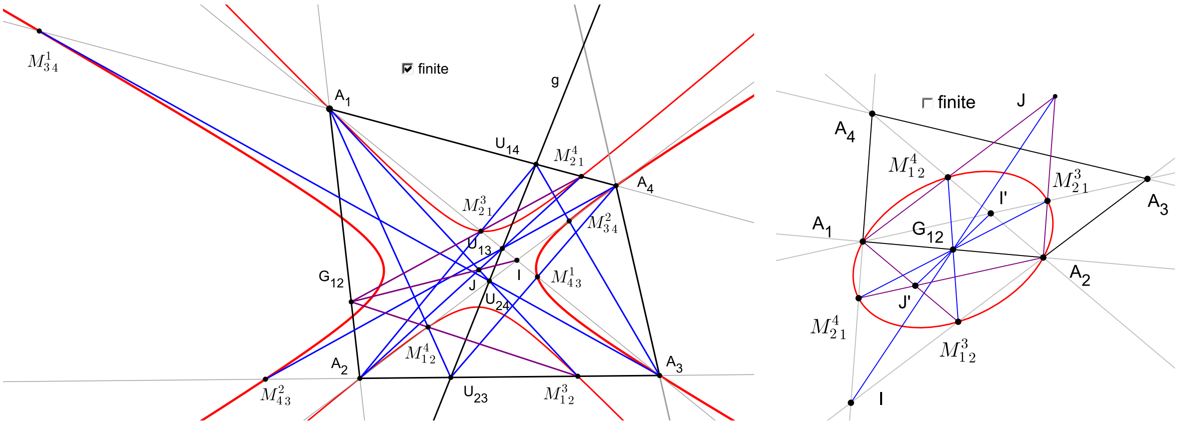

FIGURE 5. *Left* Sharygin's curves k_{12} and k_{34}, when g is a finite line. *Right* Sharygin's curve k_{12}, when g is the infinite line ω.

Proof. Let us consider the inscribed into the conic section k_{ij} complete quadrangle $A_iM_{ij}^kM_{ij}^sA_j$. Using (1), (3) and 3) in Theorem 6 we find that the diagonal triangle of $A_iM_{ij}^kM_{ij}^sA_j$ have the following vertices $U_{js} = A_iM_{ij}^k \cap A_jM_{ij}^s = A_iU_{js} \cap A_jA_s$, $U_{jk} = AM_{ij}^s \cap A_jM_{ij}^k = A_iU_{jk} \cap A_jA_k$, $G_{ij} = A_iA_j \cap M_{ij}^kM_{ij}^s$. According to Theorem 5 this triangle is self–polar. Therefore the pole of the line $g = U_{js}U_{jk}$ with respect to k_{ij} is the point G_{ij}. □

Proposition 4. *Let $(A_1A_2A_3A_4, g)$ be a (q,l)–pair. If the points A_i, A_j and A_j, A_s are two pairs of the vertices of $A_1A_2A_3A_4$ and $G_{ij} \in A_iA_j$ and $G_{js} \in A_jA_s$ are the poles of the line g with respect to Sharygin's curves k_{ij} and k_{js}, respectively, then the line $G_{ij}G_{js}$ passes through the point $U_{is} = g \cap A_iA_s$.*

Proof. Let us first introduce the notations:

$$\begin{aligned} \overline{I} = A_iA_j \cap A_sA_k \ , \quad \check{I} = A_iA_s \cap A_jA_k = I' \ , \\ \overline{J} = A_jU_{sk} \cap A_sU_{ij} \ , \quad \check{J} = A_jU_{is} \cap A_sU_{jk} \ . \end{aligned} \tag{9}$$

From Theorem 6 it follows that we can define the point G_{js} (a pole of g with respect to Sharigin's curve k_{js}) by the following way:

$$G_{js} = \overline{I}\ \overline{J} \cap A_jA_s \cap \check{I}\check{J} = \overline{I}\ \overline{J} \cap A_jA_s \cap I'\check{J}. \tag{10}$$

Let us consider the triangles $G_{ij}A_iJ'$ and $G_{js}A_s\check{J}$. Using (2), (9) and (10) we find the intersection points of the pairs of corresponding sides: $G_{ij}A_i \cap G_{js}A_s = A_iA_j \cap A_jA_s = A_j$, $G_{ij}J' \cap G_{js}\check{J} = I'J' \cap I'\check{J} = I'$, $A_iJ' \cap A_s\check{J} = A_iU_{jk} \cap A_sU_{jk} = U_{jk}$. Since they lie on the side A_jA_k of the complete quadrangle $A_1A_2A_3A_4$, the triangles $G_{ij}A_iJ'$ and $G_{js}A_s\check{J}$ are perspective and according to Theorem 2 the connecting lines of the pairs of corresponding vertices $G_{ij}G_{js}$, A_jA_s and $J'\check{J}$ are incident with one point– the perspective center. With the help of (1), (2) and (9)we get $A_iA_s \cap J'\check{J} = A_iA_s \cap A_jU_{is} = U_{is}$, from where it follows that the perspective center is the point U_{is}. Therefore the line $G_{ij}G_{js}$ passes through the point U_{is}. □

A particular case of Proposition 4 is presented on Fig. 6 for $i = 1$, $j = 2$, $s = 3$ and $k = 4$.

Proposition 4 presents us an easier technique for finding the remaining five G_{ij}–points, once we have constructed one of the six points G_{ij}.

Theorem 7. *Let $(A_1A_2A_3A_4, g)$ be a (q,l)–pair. Then the poles G_{is} of the line g with respect to the Sharygin's curves k_{is}, $i, s \in \{1,2,3,4\}$ $i \neq s$, and the diagonal points of the quadrangle $A_1A_2A_3A_4$ lie on one conic section .*

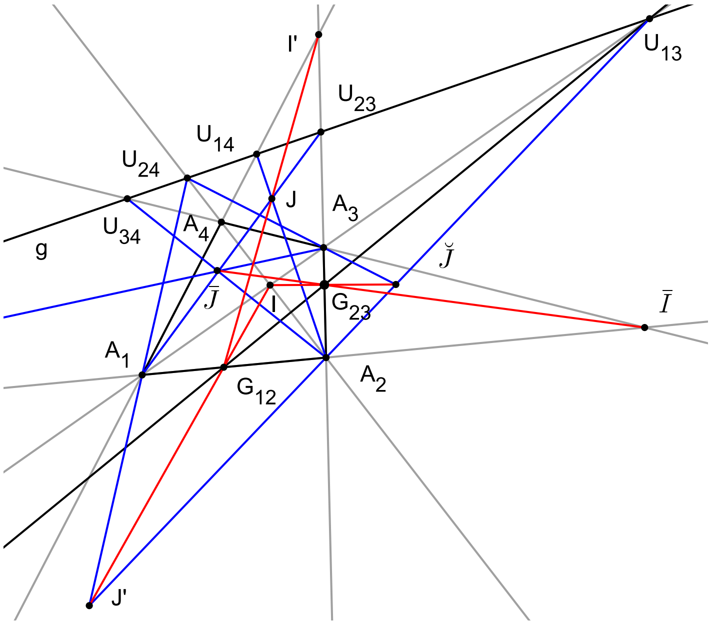

FIGURE 6. Proposition 4 for G_{12} and G_{23}

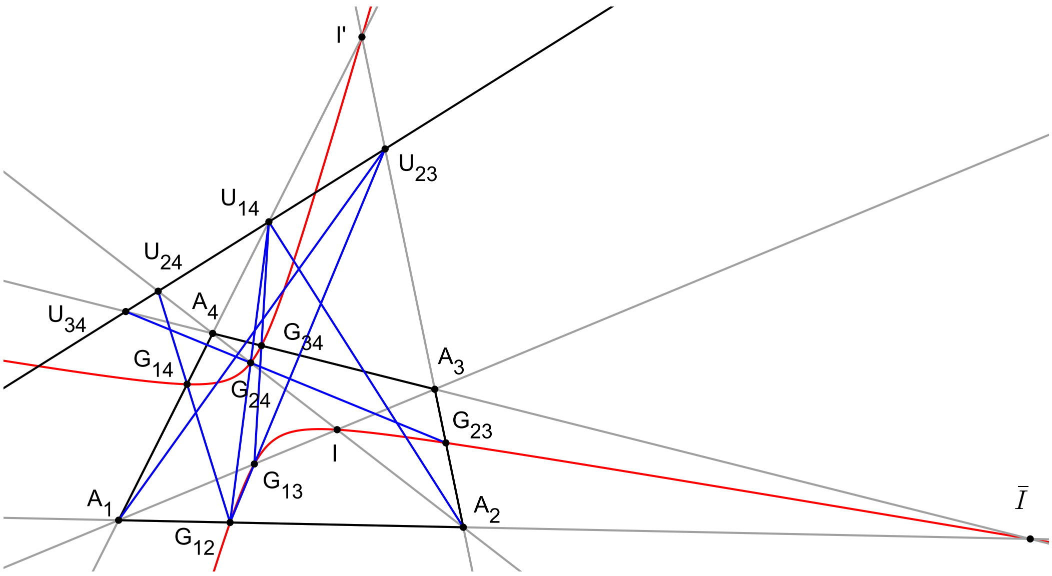

FIGURE 7. Nine points conic section

Proof. Let consider the points G_{13}, G_{23}, G_{12}, G_{24}, G_{14}, G_{34} (Fig. 7). From Proposition 4 we get that $U_{12} = G_{13}G_{23} \cap G_{24}G_{14}$, $U_{13} = G_{23}G_{12} \cap G_{14}G_{34}$ and $U_{14} = G_{12}G_{24} \cap G_{34}G_{13}$. By the condition $U_{is} \in g$ it follows that the points U_{12}, U_{13} and U_{14} are collinear. According to Theorem 3 the hexagon $G_{13}G_{23}G_{12}G_{24}G_{14}G_{34}$ is inscribed into a curve of the second power k and g is its Pascal's line.

Let consider the points G_{13}, G_{23}, I, G_{14}, G_{24}, G_{34} Applying Proposition 4 and using (2) we get the points $U_{12} = G_{13}G_{23} \cap G_{14}G_{24}$, $U_{23} = G_{23}I \cap G_{24}G_{34} = A_2A_3 \cap G_{24}U_{23}$ and $U_{14} = IG_{14} \cap G_{34}G_{13} = A_1A_4 \cap G_{34}U_{14} = U_{14}$. By the condition (1) it follows that the points U_{12}, U_{23} and U_{14} are collinear.

According to Theorem 3 the hexagon $G_{13}G_{23}IG_{14}G_{24}G_{34}$ is inscribed into a curve of second power k' and g is its Pascal's line.

The curves k and k' coincide because they have five common points.

By similar considerations for the points G_{13}, I', G_{24}, G_{34}, G_{14}, G_{12} and for the points G_{12}, $\overline{I}$, G_{34}, G_{14}, G_{24}, G_{23} it can be proved that the hexagons $G_{13}I'G_{24}G_{34}G_{14}G_{14}$ and $G_{12}\overline{I}G_{34}G_{14}G_{24}G_{23}$ are inscribed into the curve of second power k. □

Definition 5. *The curve k on which lie the nine points G_{12}, G_{13}, G_{14}, G_{23}, G_{24}, G_{34}, I, I', $\overline{I}$ will be called the curve of the nine points for the (q,l)–pair $(A_1A_2A_3A_4, g)$.*

Some properties of the curve of the nine points, when the points G_{ij} are midpoint of the Euclidian segments A_iA_j have been investigated in [4].

Theorem 8. *The pole of g with respect to the curve k of the nine points for the (q,l)–pair $(A_1A_2A_3A_4, g)$ is the point $G = G_{12}G_{34} \cap G_{13}G_{24} \cap G_{14}G_{23}$.*

Proof. Let consider the inscribed complete quadrangles $G_{12}G_{34}G_{13}G_{24}$ and $G_{12}G_{34}G_{14}G_{23}$ (Fig. 7). By Proposition 4 their diagonal triangles are $U_{14}U_{23}G$, where $G = G_{12}G_{34} \cap G_{13}G_{24}$ and $U_{13}U_{24}G'$, where $G' = G_{12}G_{34} \cap G_{14}G_{23}$, respectively. According to Theorem 5 the pole of the line $U_{14}U_{23} = g$ with respect to the curve k is the point G and the pole of the line $U_{13}U_{24} = g$ with respect to the curve k is the point G'. Since the line g is a polar of the points G and G' with respect to the curve k, then $G = G'$, which means that $G_{12}G_{34} \cap G_{13}G_{24} \cap G_{14}G_{23} = G = G'$. □

If the line g is the infinite line ω then G is the center of the nine points curve k and the G_{ij}–points are the centers of the Sharygin's curves k_{ij} for the (q,l)–pair $(A_1A_2A_3A_4, g)$.

From Theorem 6 and Proposition 4 it follows

Proposition 5. *There exists two homologies Φ and Φ' with centers O and O', respectively, for which the line g is a common axis and which are related with every pair of opposite sides for the quadrangle $A_1A_2A_3A_4$ from the (q,l)–pair $(A_1A_2A_3A_4, g)$, such that:*

$$\begin{aligned} &\Phi\ (A_i, A_j, M_{ji}^s, M_{ij}^k; J, I') = M_{ks}^j, M_{sk}^i, A_k, A_s; I, L' \ ; \\ &\Phi'(A_i, A_j, M_{ij}^s, M_{ji}^k; J', I) = M_{sk}^j, M_{ks}^i, A_k, A_s; I', L \ . \end{aligned} \tag{11}$$

Proof. Using Theorem 6, and the basic properties of any homology we can verify that the homologies $\Phi(O, g; A_i \longrightarrow M_{ks}^j)$ and $\Phi'(O, g; A_i \longrightarrow M_{sk}^j)$ have the properties (11).

Indeed from 1) in Theorem 6 it follows:

$$A_iA_j \cap M_{ks}^jM_{sk}^i = U_{ij}, \quad M_{ji}^sM_{ij}^k \cap A_kA_s = U_{ks} \ ;$$

From (2) and (4) it follows:

$$A_iM_{ji}^k \cap M_{sk}^jA_s = A_iU_{js} \cap A_jA_s = U_{js} \ ;$$

From (3) and (4) it follows:

$$A_iJ \cap M_{ks}^jI = A_iU_{js} \cap A_jA_s = U_{js} \ .$$

Taking into account and (5) we conclude that

$$\Phi(A_j, M_{ji}^s, M_{ij}^k, J) = M_{sk}^i, A_k, A_s, I.$$

From (1), (2) and (3) it follows:

$$\begin{aligned} \Phi(I') &= \Phi(A_iA_s \cap A_jA_k) = \Phi(A_iU_{is} \cap A_jU_{jk}) \\ &= M_{ks}^jU_{is} \cap M_{sk}^iU_{jk} = A_kU_{is} \cap A_sU_{jk} = L' \ . \end{aligned}$$

The proof for Φ' can be done in a similar fashion. □

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